

Coulomb integrals for the $SL(2, \mathbb{R})$ WZNW model

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Abstract

We review the Coulomb gas computation of three-point functions in the $SL(2, \mathbb{R})$ WZNW model and obtain explicit expressions for generic states. These amplitudes have been computed in the past by this and other methods but the analytic continuation in the number of screening charges required by the Coulomb gas formalism had only been performed in particular cases. After showing that ghost contributions to the correlators can be generally expressed in terms of Schur polynomials, we solve Aomoto integrals in the complex plane, a new set of multiple integrals of Dotsenko-Fateev type. We then make use of monodromy invariance to analytically continue the number of screening operators and prove that this procedure gives results in complete agreement with the amplitudes obtained from the bootstrap approach. We also compute a four-point function involving a spectral flow operator and we verify that it leads to the one unit spectral flow three-point function according to a prescription previously proposed in the literature. In addition, we present an alternative method to obtain spectral flow non-conserving n -point functions through well defined operators and we prove that it reproduces the exact correlators for $n = 3$. Independence of the result on the insertion points of these operators suggests that it is possible to violate winding number conservation modifying the background charge.

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Contents

1	Introduction	3
2	Free field realization of the $SL(2, \mathbb{R})$ WZNW model	5
2.1	Wakimoto representation	5
2.2	Vertex operators and correlation functions	7
2.3	Two- and three-point amplitudes	9
3	Spectral flow conserving three-point functions	11
3.1	Evaluation of the ghost correlator	12
3.2	Aomoto integrals in the complex plane	15
3.3	Analytic continuation	17
4	Spectral flow non-conserving three-point functions	20
4.1	Evaluation of the ghost correlator	22
4.2	Evaluation of the three-point function	24
4.3	Alternative method to compute spectral flow non-conserving amplitudes	27
5	Summary and conclusions	30
6	Appendix: useful formulas	32

1 Introduction

Several aspects of spacetime physics in string theory have been made accessible due to the development of world-sheet methods. In particular, the significant progress of algebraic techniques in rational conformal field theory (RCFT) has been crucial to improve our knowledge on string compactification and string phenomenology. Nowadays the possibility of extending the systematic understanding gained in RCFT to non-RCFT in order to describe non-compact backgrounds is under active investigation (see [1] for a complete and comprehensive review).

In this article we consider the $SL(2, \mathbb{R})$ WZNW model, a non-RCFT describing string propagation on three dimensional anti de Sitter spacetime (AdS_3). Beyond the interesting formal aspects involved in the study of non-RCFT, this model allows to address important physical issues such as curved backgrounds in string theory and the analysis of a genuine stringy regime of the AdS/CFT correspondence.

This theory has been thoroughly studied in the 90's and its current status was established in [2]–[4] where several longstanding problems were puzzled out. The Hilbert space of physical states contains all the Hermitian unitary representations of the universal cover of $\widehat{SL}(2, \mathbb{R})$. These representations were found to be the conventional unitary representations of the $SL(2, \mathbb{R})$ current algebra, namely, the principal continuous series $\widehat{\mathcal{C}}_j^\alpha$ with $j = -\frac{1}{2} + i\mathbb{R}$ and $0 \leq \alpha < 1$, and the lowest- and highest-weight principal discrete series $\widehat{\mathcal{D}}_j^\pm$, with $j \in \mathbb{R}$ and $-\frac{1}{2} < j < \frac{k-3}{2}$, and their spectral flow images $\widehat{\mathcal{C}}_j^{\alpha, w}$ and $\widehat{\mathcal{D}}_j^{\pm, w}$, respectively [2]. Here k is the level of the current algebra and the spectral flow parameter w is an integer number that can be identified with the winding number of long strings stretched close to the boundary of AdS_3 . A proof of the no-ghost theorem for this spectrum was given in [2] and verified by the exact calculation of the one loop partition function of string theory on $AdS_3 \times \mathcal{M}$ in [3], \mathcal{M} being a compact space represented by a unitary CFT on the worldsheet. To establish the consistency of the full theory, interactions have to be considered in order to verify the closure of the operator product expansion. Important progress has been achieved in the resolution of this problem in [4] where some correlation functions were computed and the structure of the factorization of a four-point function was shown to agree with the physical spectrum of the theory. However a complete proof of unitarity, involving arbitrary spectral flow sectors, is still lacking.

Various correlators are known in this theory and its euclidean version, the H_3^+ or $SL(2, \mathbb{C})/SU(2)$ coset model. The structure constants of the H_3^+ model were obtained in [5] using a generalization of the bootstrap approach. The analytic continuation of these results to the $SL(2, \mathbb{R})$ WZNW model was discussed in [4] where two- and three-point functions involving spectral flowed states were also computed. The methods followed in these works rely on the invariance of correlation functions under the Lie and conformal symmetries, which leads to the Knizhnik-Zamolodchikov (KZ) equations, and they also take advantage of the properties of the so-called spectral flow operator, an auxiliary degenerate operator interpolating between different w sectors, which gives additional differential equations to be satisfied by the correlators. Actually, a suitable number of these spectral flow operators has to be inserted in the correlation functions in order to transfer winding to the physical states. For instance the four-point function of three generic states and one spectral flow operator was computed in [4] in order to obtain the one unit spectral flow three-point function¹. The same method was used in [6, 7] to compute the five-point function involving three generic states and two spectral flow operators, leading to the spectral flow conserving three-point function of two $w = 1$ and one $w = 0$ states².

The problem of constructing a complete set of solutions to the KZ equations for generic four-point functions has not been solved yet. These equations cannot be reduced to a system of ordinary differential equations [8] as in the case of the compact $SU(2)$ WZNW model [9] and thus the bootstrap approach does

¹A well known feature of string interactions is that winding number conservation may be violated according to a precise pattern determined by the properties of $SL(2, \mathbb{R})$ representations [4, 6].

²Recall that w takes positive integer values in the x -basis [4].

not lie on a firm rigorous mathematical basis up to now. Solutions have been found only as formal power series in the cross-ratio on the worldsheet [4, 8, 10]. Aiming to explore an alternative method to compute four-point functions, in this article we consider the Coulomb gas formalism³. This approach appears to be ad-hoc and not very well grounded, since there are no singular vectors in the unitary spectrum of $\widehat{SL}(2, \mathbb{R})$ while it is well known that degenerate fields provide the formal mathematical foundation for the background charge method [13]. Some skepticism on this procedure was raised in [14], where the Hilbert space of physical states of string theory on AdS_3 was constructed as the BRST cohomology on the Fock spaces of free fields and it was shown to differ from the spectrum determined in [2]. From a computational perspective, one disadvantage of this method is that it is implemented in the necessarily more cumbersome m -basis. However, while only one unit spectral flow operators have been constructed in the x -basis [4], it is only in the m -basis that the spectral flow operation can be performed in arbitrary winding sectors so far.

The Coulomb gas method was used in reference [15] to compute two- and three-point functions⁴. It was later extended to include winding and to obtain spectral flow non-conserving three-point functions in [18, 19]. The starting point of the procedure implemented in [15, 19] involved three-point functions preserving winding number conservation with at least one highest-weight state, whereas the one unit spectral flow amplitudes included at least two highest-weight states. The highest-weight condition was necessary in order to simplify the computation and manage to solve it explicitly. One of the difficulties comes from the $\beta - \gamma$ ghost fields required by the Wakimoto realization and the fact that the underlying representations of $SL(2, \mathbb{R})$ are infinite dimensional. This leads to consider arbitrary numbers of screening currents and intricate ghost correlators. Another trouble of this approach arises in the values required by unitarity for the spin j and the necessity to analytically continue to non-integer numbers of screening operators. The continuation performed in [15] leads to the exact results when the highest-weight condition is relaxed to include an arbitrary global descendant in the correlator [20], but so far it has not been shown that the most general case can be obtained in this way. Similarly, the spectral flow non-conserving three-point function computed in [19] reproduces the exact result when it involves at least two highest-weight states.

In this article, we are able to compute the required expectation values of the $\beta - \gamma$ system for generic three-point functions and for a particular four-point function, using standard bosonization, the definition of the Vandermonde determinant and Schur polynomials. This leads to multiple integral expressions for unrestricted three-point functions which we manage to solve working out Aomoto integral [21] in the complex plane, a new integral formula of Dotsenko-Fateev type [22, 23]. We then make use of monodromy invariance to analytically continue to non-integer numbers of screening operators and show that the results obtained in this way are in full agreement with those of [4, 5, 7] for generic spectral flow conserving three-point functions. We also reproduce the computation of the one unit spectral flow three-point function performed in [6, 4] using free fields, and again we show full agreement with the exact results. This problem involves the evaluation of a four-point function including one spectral flow operator, which we manage to do extending the techniques discussed above for the winding number conserving case. Thus we demonstrate the complete coincidence between correlation functions computed in the free field formalism and all the known exact results. Additionally, we present an alternative method to obtain winding number non-preserving n -point functions in the Coulomb gas approach through well defined operators. The independence of the result on the insertion points of these operators suggests that it is possible to violate winding number conservation modifying the background charge. We prove that this is in fact the case for $n = 3$.

³ Another method, which exploits the relation between the H_3^+ and Liouville theories, was presented in [11, 12].

⁴See [16] for Coulomb gas computations of $SL(2)$ correlators involving states in admissible representations and [17] for more recent work on the related Liouville theory.

This work is organized as follows. In order to set up our notation, we include a brief review of the free field realization of $SL(2, \mathbb{R})$ in Section 2. The computation of winding conserving three-point functions is given in Section 3. This section contains three parts: the resolution of the $\beta - \gamma$ correlator, the explicit evaluation of Aomoto integrals in the complex plane and finally, the analytic continuation to non-integer numbers of screening operators. Comparison with previous results obtained through the bootstrap approach is presented as well. In Section 4 we compute the spectral flow non-conserving three-point function using the Coulomb gas method following the prescription developed in [6, 4]. We also present a novel method to compute the winding number non-conserving n -point functions and we verify that it reproduces the exact results for $n = 3$. Section 5 offers conclusions and a discussion about perspectives for the extension of the techniques used throughout the text to generic four-point functions.

2 Free field realization of the $SL(2, \mathbb{R})$ WZNW model

Mainly in order to set up our notation and conventions we will briefly review in this section the free field realization of the $SL(2, \mathbb{R})$ WZNW model and the computation of correlation functions in this approach.

2.1 Wakimoto representation

Bosonic string propagation in an AdS_3 background is described by a non-linear sigma model equivalent to a WZNW model on $SL(2, \mathbb{R})$ (actually, its Euclidean version on $SL(2, \mathbb{C})/SU(2)$). The action of the theory is given by

$$S = \frac{k}{8\pi} \int d^2z (\partial\phi\bar{\partial}\phi + e^{2\phi}\bar{\partial}\gamma\partial\bar{\gamma}), \quad (2.1)$$

where $\{\phi, \gamma, \bar{\gamma}\}$ are the Poincaré coordinates of the (euclidean) AdS_3 spacetime and $k = l^2/l_s^2$, l being related to the scalar curvature of AdS_3 as $\mathcal{R} = -2/l^2$ and l_s being the fundamental string length. While ϕ is a real field, $\{\gamma, \bar{\gamma}\}$ are complex coordinates parameterizing the boundary of AdS_3 , which is located at $\phi \rightarrow \infty$.

Eq. (2.1) can be obtained by integrating out the one-form auxiliary fields β and $\bar{\beta}$ in the following action:

$$S = \frac{1}{4\pi} \int d^2z \left[\partial\phi\bar{\partial}\phi - \frac{2}{\alpha_+} \mathcal{R}^{(2)}\phi + \beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma} - \beta\bar{\beta}e^{-\frac{2}{\alpha_+}\phi} \right], \quad (2.2)$$

where $\alpha_+ = \sqrt{2(k-2)}$, $\mathcal{R}^{(2)}$ is the scalar curvature of the worldsheet and k -dependent renormalization factors have been included after quantizing [24, 25].

The linear dilaton term in (2.2) can be interpreted as the effect of a background charge at infinity. In fact, the large ϕ region can be explored treating the interaction, perturbatively, as a screening charge. In this limit, the theory reduces to a free linear dilaton field ϕ and a free $\beta - \gamma$ system with propagators given by the following OPE:

$$\phi(z)\phi(0) \sim -\ln z, \quad (2.3)$$

$$\beta(z)\gamma(0) \sim \frac{1}{z}. \quad (2.4)$$

Similar expressions hold for the antiholomorphic content. In what follows we will focus our discussion on the left moving component of the theory and we will assume that all the steps go through to the right moving part as well, indicating the left-right matching condition only if necessary.

Generic correlation functions would not be expected to be reliable in this approximation. Notice that the original action (2.1) has a singularity in this limit. However we will demonstrate below that the

free field computation of two- and three-point functions gives results in full agreement with the exact calculations. We will show that this is the case even when winding number conservation is violated and the computation of a four-point function is necessarily involved.

The theory is invariant under the action of two copies of the $SL(2, \mathbb{R})$ current algebra at level k . We will denote the currents generating this algebra by J^a with $a = \pm, 3$. These currents are given in the Wakimoto representation by

$$J^+ = \beta, \quad (2.5)$$

$$J^3 = -\beta\gamma - \frac{\alpha_+}{2} \partial\phi, \quad (2.6)$$

$$J^- = \beta\gamma^2 + \alpha_+\gamma\partial\phi + k\partial\gamma, \quad (2.7)$$

and they verify the following OPE:

$$J^+(z)J^-(0) \sim \frac{k}{z^2} - \frac{2J^3(0)}{z}, \quad (2.8)$$

$$J^3(z)J^\pm(0) \sim \pm \frac{J^\pm(0)}{z}, \quad (2.9)$$

$$J^3(z)J^3(0) \sim -\frac{k/2}{z^2}, \quad (2.10)$$

in agreement with the $SL(2, \mathbb{R})$ current algebra at level k commutation relations.

The Sugawara construction gives rise to the following energy-momentum tensor:

$$T_{SL(2, \mathbb{R})} = -\frac{1}{2} \partial\phi\partial\phi - \frac{1}{\alpha_+} \partial^2\phi + \beta\partial\gamma, \quad (2.11)$$

which leads to a Virasoro algebra with central charge $c = 3k/(k-2)$.

In the next section we will find that it is convenient to bosonize the $\beta - \gamma$ system. We do it in the standard way, i.e., after introducing free bosonic fields u and v with OPE

$$u(z)u(0) = v(z)v(0) \sim -\ln z, \quad (2.12)$$

we parametrize the $\beta - \gamma$ system as follows:

$$\beta = -i\partial v e^{-u+iv}, \quad (2.13)$$

$$\gamma = e^{u-iv}. \quad (2.14)$$

The currents take now the following form:

$$J^+ = -i\partial v e^{-u+iv}, \quad (2.15)$$

$$J^3 = \partial u - \frac{\alpha_+}{2} \partial\phi, \quad (2.16)$$

$$J^- = e^{u-iv}[(k-2)\partial u - i(k-1)\partial v + \alpha_+\partial\phi], \quad (2.17)$$

while the energy-momentum tensor is given by

$$T_{SL(2, \mathbb{R})} = -\frac{1}{2} \partial u \partial u - \frac{1}{2} \partial v \partial v - \frac{1}{2} \partial\phi \partial\phi - \frac{i}{2} \partial^2 u + \frac{i}{2} \partial^2 v - \frac{1}{\alpha_+} \partial^2\phi. \quad (2.18)$$

2.2 Vertex operators and correlation functions

The Hilbert space of physical states for this model was established in [2]. It contains all the Hermitian unitary representations of the universal cover of $\widehat{SL}(2, \mathbb{R})$. The conventional unitary representations of the current algebra, namely, the principal continuous series $\hat{\mathcal{C}}_j^{\alpha,0}$ with $j = -1/2 + i\lambda$, $\lambda \in \mathbb{R}$ and $0 \leq \alpha < 1$, and the lowest- and highest-weight principal discrete series $\hat{\mathcal{D}}_j^{\pm,0}$, with $j \in \mathbb{R}$ are, of course, among those included in the spectrum. In addition, their spectral flow images $\hat{\mathcal{C}}_j^{\alpha,w}$ and $\hat{\mathcal{D}}_j^{\pm,w}$ must be taken into account. These series are related to those with $w = 0$ by the spectral flow automorphism of the current algebra, i.e., the symmetry under which the modes of the currents transform as

$$J_n^\pm \rightarrow \tilde{J}_n^\pm = J_{n \pm w}^\pm, \quad (2.19)$$

$$J_n^3 \rightarrow \tilde{J}_n^3 = J_n^3 - \frac{k}{2} w \delta_{n,0}. \quad (2.20)$$

As it was pointed out in [2], in standard models based on compact Lie groups the spectral flow does not generate new types of representations, but for the $SL(2, \mathbb{R})$ WZNW model representations with different amounts of w are, in general, inequivalent. A relevant exception is the case of the series $\hat{\mathcal{D}}_j^{\pm, \mp 1}$ and $\hat{\mathcal{D}}_{\frac{k}{2}-2-j}^{\mp, 0}$. The equivalence of these representations has an important consequence on the allowed values of j , namely, for discrete representations one has

$$-\frac{1}{2} < j < \frac{k-3}{2}. \quad (2.21)$$

The full Hilbert space of the theory is, then,

$$\mathcal{H} = \bigoplus_{w=-\infty}^{+\infty} \left\{ \left[\int_{-\frac{1}{2}}^{\frac{k-3}{2}} dj \hat{\mathcal{D}}_j^{+,w} \otimes \hat{\mathcal{D}}_j^{+,w} \right] \oplus \left[\int_{-\frac{1}{2}+i\mathbb{R}}^1 dj \int_0^1 d\alpha \hat{\mathcal{C}}_j^{\alpha,w} \otimes \hat{\mathcal{C}}_j^{\alpha,w} \right] \right\}, \quad (2.22)$$

where $\hat{\mathcal{D}}_j^{+,w}$ is the irreducible representation of the $SL(2, \mathbb{R})$ current algebra generated from the highest-weight state $|j; w\rangle$ defined by

$$J_{n+w}^+ |j; w\rangle = J_{n-w-1}^- |j; w\rangle = J_n^3 |j; w\rangle = 0, \quad (n = 1, 2, \dots) \quad (2.23)$$

$$J_0^3 |j; w\rangle = \left(j + \frac{k}{2} w \right) |j; w\rangle, \quad (2.24)$$

$$\left[-\left(J_0^3 - \frac{k}{2} w \right)^2 + \frac{1}{2} (J_w^+ J_{-w}^- + J_{-w}^- J_w^+) \right] |j; w\rangle = -j(j+1) |j; w\rangle, \quad (2.25)$$

and $\hat{\mathcal{C}}_j^{\alpha,w}$ is generated from the state $|j, \alpha; w\rangle$ obeying

$$J_{n \pm w}^\pm |j, \alpha; w\rangle = J_n^3 |j, \alpha; w\rangle = 0, \quad (2.26)$$

$$J_0^3 |j, \alpha; w\rangle = \left(\alpha + \frac{k}{2} w \right) |j, \alpha; w\rangle, \quad (2.27)$$

$$\left[-(J_0^3 - \frac{k}{2}w)^2 + \frac{1}{2} (J_w^+ J_{-w}^- + J_{-w}^- J_w^+) \right] |j, \alpha; w\rangle = -j(j+1) |j, \alpha; w\rangle. \quad (2.28)$$

A suitable representation of the vertex operators creating these states was introduced in the discrete light cone approach in [14]. In terms of the fields u and v , these vertex operators can be written as

$$V_{j,m,\bar{m}}^w = e^{(j-m-w)u(z)} e^{(j-\bar{m}-w)u(\bar{z})} e^{-i(j-m)v(z)} e^{-i(j-\bar{m})v(\bar{z})} e^{\left(\frac{2}{\alpha_+}j + \frac{\alpha_+}{2}w\right)\phi(z,\bar{z})}, \quad (2.29)$$

where we are assuming, as usual, that $(m - \bar{m})$ is an integer number.

It is straightforward to check the following properties:

$$\tilde{J}^\pm(z) V_{j,m,\bar{m}}^w(0) \sim \frac{\pm j - m}{z} V_{j,m\pm 1,\bar{m}}^w(0), \quad (2.30)$$

$$\tilde{J}^3(z) V_{j,m,\bar{m}}^w(0) \sim \frac{m}{z} V_{j,m,\bar{m}}^w(0), \quad (2.31)$$

in agreement with (2.23)–(2.28). Notice that the zero modes of the currents have been shifted according to the spectral flow sector of the vertex operators, i.e., the shift (2.19)–(2.20) cancels the factors $z^{\pm w}$ and $kw/2z$ in the OPE.

The operators (2.29) reduce to the well known vertices [26]

$$V_{j,m,\bar{m}} = \gamma^{j-m} \bar{\gamma}^{j-\bar{m}} e^{2j\phi/\alpha_+} \quad (2.32)$$

in the case $w = 0$. For future reference, it is convenient to recall here that these operators appear when looking for the asymptotic expressions of the following normalizable operators in the $SL(2, \mathbb{C})/SU(2)$ model

$$\Phi_j(x, \bar{x}; z, \bar{z}) = \frac{1+2j}{\pi} \left(|\gamma - x|^2 e^{\phi/\alpha_+} + e^{-\phi/\alpha_+} \right)^{2j}, \quad (2.33)$$

where x, \bar{x} keep track of the $SL(2, \mathbb{C})$ quantum numbers. These operators, when transformed to the m -basis through the following Fourier integral:

$$\Phi_{j,m,\bar{m}}(z, \bar{z}) = \int d^2x x^{j-m} \bar{x}^{j-\bar{m}} \Phi_{-1-j}(x, \bar{x}, z, \bar{z}) \quad , \quad (2.34)$$

give, in the limit $\phi \rightarrow \infty$,

$$\Phi_{j,m,\bar{m}}(z, \bar{z}) \sim V_{j,m,\bar{m}} + \frac{1+2j}{\pi} c_{m,\bar{m}}^{-1-j} V_{-1-j,m,\bar{m}}, \quad (2.35)$$

where

$$c_{m,\bar{m}}^j = \pi \gamma (1+2j) \frac{\Gamma(-j+m)\Gamma(-j-\bar{m})}{\Gamma(1+j+m)\Gamma(1+j-\bar{m})} \quad (2.36)$$

and

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}. \quad (2.37)$$

While both terms in (2.35) contribute if the state is in a principal continuous series, Eq. (2.35) reduces to the vertex (2.32) when the state belongs to a highest or lowest-weight representation.

We can introduce winding number in (2.35) and define asymptotically

$$\Phi_{j,m,\bar{m}}^w(z, \bar{z}) \sim V_{j,m,\bar{m}}^w + \frac{1+2j}{\pi} c_{m,\bar{m}}^{-1-j} V_{-1-j,m,\bar{m}}^w. \quad (2.38)$$

Recall that (2.35) is a classical expression. The second term requires an additional factor to account for quantum corrections [5] which is also necessary in (2.38).

All relevant correlators will involve vertex operators like (2.29). In the Coulomb gas formalism it is mandatory to insert, in addition, some operators in order to screen the charges of these vertices and the background charge. The so-called screening operators in the $SL(2, \mathbb{R})$ WZNW model are the following [26]:

$$\mathcal{S}_+ = \int d^2y \beta(y) \bar{\beta}(\bar{y}) e^{-2\phi(y, \bar{y})/\alpha_+}, \quad (2.39)$$

$$\mathcal{S}_- = \int d^2y [\beta(y) \bar{\beta}(\bar{y})]^{k-2} e^{-\alpha_+ \phi(y, \bar{y})}, \quad (2.40)$$

where the screening currents are defined so that they have trivial OPE ^{5, 6} with $J^{3,\pm}$.

Summarizing, in the context of the Coulomb gas formalism one has to compute expectation values of the form

$$\mathcal{A}_n = \langle V_{j_1, m_1, \bar{m}_1}^{w_1}(z_1, \bar{z}_1) \cdots V_{j_n, m_n, \bar{m}_n}^{w_n}(z_n, \bar{z}_n) \mathcal{S}_+^{s_+} \mathcal{S}_-^{s_-} \rangle, \quad (2.41)$$

where the number of screening operators is determined from the following conservation laws [15]:

$$\sum_i \alpha_i^\phi = -\frac{2}{\alpha_+}, \quad \sum_i \alpha_i^u = -1, \quad \sum_i \alpha_i^v = i, \quad (2.42)$$

$\alpha^{\phi, u, v}$ being the charges of the fields ϕ, u, v , respectively. ⁷

Notice that \mathcal{S}_+ is the interaction term in (2.2), and therefore computing amplitudes with $s_- = 0$ is completely equivalent to a perturbative expansion of order s_+ in the path integral formalism. Moreover, as we have already seen, unitarity requires $-\frac{1}{2} < j < \frac{k-3}{2}$ for the principal discrete series or $j = -\frac{1}{2} + i\lambda$ with $\lambda \in \mathbb{R}$ for the continuous ones. It is then necessary to consider $s_+, s_- \notin \mathbb{Z}^+$, i.e., after computing (2.41) for an integer number of screening insertions it is necessary to continue this function to non-integer values of s_+ and s_- . Actually, once this generalization is allowed any correlator can be computed using only one kind of screening operators. In what follows we will consider only screening operators of type \mathcal{S}_+ and we will write \mathcal{S} instead of \mathcal{S}_+ , and s instead of s_+ .

2.3 Two- and three-point amplitudes

The two- and three-point functions in the $SL(2, \mathbb{C})/SU(2)$ WZNW model were computed exactly in [5] and winding number was included in [4]. For future reference we quote here the results.

$$\begin{aligned} \langle \Phi_{j,m,\bar{m}}^w(z) \Phi_{j',m',\bar{m}'}^{w'}(z') \rangle &= \delta^2(m+m') \delta_{w+w'}(z-z')^{-2\hat{\Delta}} (\bar{z}-\bar{z}')^{-2\bar{\Delta}} \\ &\times \left[\delta(j+j'+1) + B(1+j) c_{m,\bar{m}}^{-1-j} \delta(j-j') \right], \end{aligned} \quad (2.43)$$

⁵Recall that there is a total derivative in the OPE with J^- which requires a careful treatment of contact terms [15].

⁶The relation between \mathcal{S}_+ and the zero momentum mode of the dilaton was established in [27].

⁷ Spectral flow operators must be also considered if winding number conservation is not preserved. We will discuss this subject in section 4.

where

$$\hat{\Delta} = \Delta(j) - wm - \frac{k}{4}w^2 = -\frac{j(j+1)}{k-2} - wm - \frac{k}{4}w^2, \quad (2.44)$$

$$\delta^2(m) \equiv \int d^2x x^{m-1} \bar{x}^{\bar{m}-1} = 4\pi^2 \delta(m + \bar{m}) \delta_{m-\bar{m}}, \quad (2.45)$$

$$B(j) = -\frac{1}{\pi\rho} \frac{\nu^{1-2j}}{\gamma(\rho(1-2j))}, \quad (2.46)$$

with ν given by

$$\nu = -\frac{\pi}{\rho} \frac{1}{\gamma(-\rho)}, \quad (2.47)$$

and we have defined $\rho = -1/(k-2)$. This value of ν was set in [5] by requiring consistency between the two- and three-point functions but its choice will not affect the discussion in the rest of the paper.

The Coulomb gas computation of the two-point function was performed in reference [15] for $m_i = \bar{m}_i$ and winding number was included in reference [19] where the expression (2.43) was obtained.

It is well known that winding number conservation may be violated in three and higher-point functions [6, 4]. If we assume that winding number is conserved, i.e.,

$$w_1 + w_2 + w_3 = 0, \quad (2.48)$$

a generic three-point function has the following expression:

$$\left\langle \prod_{i=1}^3 \Phi_{j_i, m_i, \bar{m}_i}^{w_i}(z_i) \right\rangle = \delta^2(\sum m_i) C(1+j_i) W(j_i, m_i, \bar{m}_i) \prod_{i < j} |z_{ij}|^{-2\Delta_{ij}}, \quad (2.49)$$

where $z_{ij} = z_i - z_j$, $\Delta_{12} = \Delta_1 + \Delta_2 - \Delta_3$, etc.

The function $C(j_i)$ is the three-point function of the primaries in the x -basis computed in [5]. It reads

$$C(j_i) = -\frac{G(1 - \sum j_i) G(-j_{12}) G(-j_{13}) G(-j_{23})}{2\pi^2 \nu^{j_1+j_2+j_3-1} \gamma\left(\frac{k-1}{k-2}\right) G(-1) G(1-2j_1) G(1-2j_2) G(1-2j_3)}, \quad (2.50)$$

where

$$G(j) = (k-2)^{\frac{j(k-1-j)}{2(k-2)}} \Gamma_2(-j|1, k-2) \Gamma_2(k-1-j|1, k-2), \quad (2.51)$$

$\Gamma_2(x|1, w)$ being the Barnes double Gamma function and $j_{12} = j_1 + j_2 - j_3$, etc.

The function $W(j_i, m_i, \bar{m}_i)$ is given by the following integral:

$$\begin{aligned} W(j_i, m_i, \bar{m}_i) &= \int d^2x_1 d^2x_2 x_1^{j_1-m_1} \bar{x}_1^{j_1-\bar{m}_1} x_2^{j_2-m_2} \bar{x}_2^{j_2-\bar{m}_2} \\ &\times |1-x_1|^{-2j_{13}-2} |1-x_2|^{-2j_{23}-2} |x_1-x_2|^{-2j_{12}-2}, \end{aligned} \quad (2.52)$$

which was computed in [28].

The one unit spectral flow three-point function was first evaluated in [4]. It is given by

$$\left\langle \prod_{i=1}^3 \Phi_{j_i, m_i, \bar{m}_i}^{w_i}(z_i) \right\rangle = \delta^2 \left(\sum m_i + \frac{k}{2} \right) \frac{\tilde{C}(1+j_i) \tilde{W}(j_i, m_i, \bar{m}_i)}{\gamma(j_1+j_2+j_3+3-\frac{k}{2})} \prod_{i < j} z_{ij}^{-\hat{\Delta}_{ij}} \bar{z}_{ij}^{-\bar{\Delta}_{ij}}, \quad (2.53)$$

where now ⁸

$$w_1 + w_2 + w_3 = 1, \quad (2.54)$$

$$\tilde{C}(j_i) \sim B(j_1) C(\frac{k}{2} - j_1, j_2, j_3), \quad (2.55)$$

$$\tilde{W}(j_i, m_i, \bar{m}_i) = \frac{\Gamma(1+j_1+m_1)}{\Gamma(-j_1-\bar{m}_1)} \frac{\Gamma(1+j_2+m_2)}{\Gamma(-j_2-\bar{m}_2)} \frac{\Gamma(1+j_3+\bar{m}_3)}{\Gamma(-j_3-m_3)}, \quad (2.56)$$

and $\hat{\Delta}_{ij}$ equals Δ_{ij} with Δ_1 replaced by $\Delta_1 + m_1 - k/4$.

The computation of three-point functions of $w = 0$ states was performed in the Coulomb gas approach for correlators with at least one insertion in the principal discrete series in [15]. States in other winding sectors were considered in [19], where winding number conserving three-point functions with at least one highest-weight state as well as winding number non-conserving three-point functions with at least two highest-weight states were computed. As we have already mentioned the expressions in the Coulomb gas approach agree with the exact ones. However, the highest-weight condition greatly simplifies the computations and a complete proof of the agreement is not available yet. So in the forthcoming sections we will develop techniques which will allow us to obtain the full expressions (2.49) and (2.53) using free fields.

3 Spectral flow conserving three-point functions

In this section we obtain the spectral flow conserving three-point functions for generic states in the Coulomb gas approach. As we have already said, we need to compute the following expression:

$$\mathcal{A}_3 = \langle V_{j_1, m_1, \bar{m}_1}^{w_1}(z_1) V_{j_2, m_2, \bar{m}_2}^{w_2}(z_2) V_{j_3, m_3, \bar{m}_3}^{w_3}(z_3) \mathcal{S}^s \rangle, \quad (3.1)$$

where s is determined from the conservation laws, which in this case lead to

$$\sum_{i=1}^3 j_i + 1 = s, \quad (3.2)$$

$$\sum_{i=0}^3 m_i = \sum_{i=0}^3 \bar{m}_i = 0, \quad (3.3)$$

$$\sum_{i=1}^3 w_i = 0. \quad (3.4)$$

⁸Recall that our convention for m , and consequently for w , differs by a sign from that in [4].

The expectation value (3.1) involves a ϕ -dependent part and a ghost correlator, i.e., a u - v -dependent part. The first one is trivial and gives the following Coulomb integrals:

$$\Gamma(-s) \int \prod_{i=1}^s d^2 y_i \prod_{k=1}^3 |z_k - y_i|^{-4\rho j_k + 2w_k} \prod_{i < j}^s |y_i - y_j|^{4\rho}, \quad (3.5)$$

where the factor $\Gamma(-s)$ is the contribution of the zero modes. The ghost correlator does depend upon the y_i 's and it must be included into the integrand of (3.5). It can be written as

$$\left\langle \prod_{l=1}^3 e^{(j_l - m_l - w_l)u(z_l)} \prod_{i=1}^s e^{-u(y_i)} \right\rangle \left\langle \prod_{l=1}^3 e^{-i(j_l - m_l)v(z_l)} \prod_{i=1}^s \partial_i e^{iv(y_i)} \right\rangle = \prod_{i=1}^s \prod_{l=1}^3 (z_l - y_i)^{-w_l} \mathcal{P}^{-1} \partial_1 \cdots \partial_s \mathcal{P}, \quad (3.6)$$

where

$$\mathcal{P} = \prod_{i=1}^s \prod_{l=1}^3 (z_l - y_i)^{m_l - j_l} \prod_{i < j} (y_i - y_j). \quad (3.7)$$

Here we have only quoted the holomorphic component; the antiholomorphic part has the same form, with the replacement $m_l \rightarrow \bar{m}_l$.

Expressions (3.5)–(3.7) simplify when $(z_1, z_2, z_3) = (0, 1, +\infty)$. Indeed, in this case (3.5) reads

$$\Gamma(-s) \int \prod_{i=1}^s d^2 y_i |y_i|^{-4\rho j_1 + 2w_1} |1 - y_i|^{-4\rho j_2 + 2w_2} \prod_{i < j}^s |y_i - y_j|^{4\rho}, \quad (3.8)$$

and (3.7) is

$$\mathcal{P} = \prod_{i=1}^s y_i^{m_1 - j_1} (1 - y_i)^{m_2 - j_2} \prod_{i < j} (y_i - y_j). \quad (3.9)$$

Notice that the dependence on the winding number cancels when combining (3.8) and (3.6). This is an explicit verification of the observation made in [4, 11] that the coordinate independent part of the correlation functions in the m -basis depend on the sum $\sum_i w_i$.

The ghost contribution to the three-point function was evaluated in [15] for the specific case in which all $m_l = \bar{m}_l$, $w_l = 0$ and the state created by $V_{j_1, m_1, \bar{m}_1}^{w_1}$ is a highest-weight state, namely, $m_1 = \bar{m}_1 = j_1$. We now compute it for three generic states.

3.1 Evaluation of the ghost correlator

Equation (3.9) can be rewritten, up to an irrelevant sign, in terms of the Vandermonde determinant, i.e.,

$$\mathcal{P} = \left[\prod_{i=1}^s y_i^{m_1 - j_1} (1 - y_i)^{m_2 - j_2} \right] \det \left(y_i^{j-1} \right), \quad (3.10)$$

and then

$$\mathcal{P} = \det \left[y_i^{j-j_1 + m_1 - 1} (1 - y_i)^{m_2 - j_2} \right]. \quad (3.11)$$

Since each row in this determinant depends upon a single variable, the multiple derivatives in equation (3.6) can be computed row by row with only one derivation, namely,

$$\partial_1 \cdots \partial_s \mathcal{P} = \det \left\{ \partial_i \left[y_i^{j-j_1 + m_1 - 1} (1 - y_i)^{m_2 - j_2} \right] \right\}. \quad (3.12)$$

Performing the derivatives in this last determinant we get

$$\partial_1 \cdots \partial_s \mathcal{P} = \left[\prod_{i=1}^s y_i^{m_1-j_1-1} (1-y_i)^{m_2-j_2-1} \right] \det \left(\ell_j^0 y_i^{j-1} + \ell_j^1 y_i^j \right), \quad (3.13)$$

with

$$\ell_j^0 = j - j_1 + m_1 - 1, \quad (3.14)$$

$$\ell_j^1 = 1 - j + j_1 - m_1 + j_2 - m_2. \quad (3.15)$$

From here we obtain

$$\mathcal{P}^{-1} \partial_1 \cdots \partial_s \mathcal{P} = \left[\prod_{i=1}^s y_i^{-1} (1-y_i)^{-1} \right] \frac{\det \left(\ell_j^0 y_i^{j-1} + \ell_j^1 y_i^j \right)}{\det(y_i^{j-1})}. \quad (3.16)$$

The determinant in the numerator of this equation may be computed performing the multiple distributions and noticing that the only non-vanishing contributions come from those determinants in which the columns have all different powers. Therefore,

$$\det \left(\ell_j^0 y_i^{j-1} + \ell_j^1 y_i^j \right) = \sum_{n=0}^s \left[\prod_{k=1}^s \ell_k^{\lambda_{s+1-k}^n} \right] \det(y_i^{j-1+\lambda_{s+1-j}^n}), \quad (3.17)$$

where for every $n = 1, 2, \dots, s$ we have introduced the partition

$$\lambda^n = \underbrace{(1, 1, \dots, 1)}_{n \text{ entries}}, \quad (3.18)$$

and we have set $\lambda_k^n = 0$ when $k > n$ and $\lambda^0 = 0$.

Using that

$$\ell_1^0 \cdots \ell_{s-n}^0 = \frac{\Gamma(m_1 - j_1 + s - n)}{\Gamma(m_1 - j_1)}, \quad (3.19)$$

and

$$\ell_{s-n+1}^1 \cdots \ell_s^1 = \frac{\Gamma(n + 1 - s + j_1 - m_1 + j_2 - m_2)}{\Gamma(1 - s + j_1 - m_1 + j_2 - m_2)}, \quad (3.20)$$

we finally obtain

$$\begin{aligned} \mathcal{P}^{-1} \partial_1 \cdots \partial_s \mathcal{P} &= \left[\prod_{i=1}^s y_i^{-1} (1-y_i)^{-1} \right] \sum_{n=0}^s \frac{\Gamma(m_1 - j_1 + s - n)}{\Gamma(m_1 - j_1)} \\ &\quad \times \frac{\Gamma(n + 1 - s + j_1 - m_1 + j_2 - m_2)}{\Gamma(1 - s + j_1 - m_1 + j_2 - m_2)} \frac{\det \left(y_i^{j-1+\lambda_{s+1-j}^n} \right)}{\det(y_i^{j-1})}. \end{aligned} \quad (3.21)$$

Notice that (3.19) vanishes when $m_1 = j_1$, i.e., when $V_{j_1, m_1, \bar{m}_1}^{w_1}$ creates a highest-weight state, unless $n = s$. In that case only one term of the above sum survives and the computation reduces to the one performed in [15] with the following result:

$$\mathcal{P}^{-1} \partial_1 \cdots \partial_s \mathcal{P} = \frac{\Gamma(1 - j_2 - m_2)}{\Gamma(-j_3 + m_3)} \prod_{i=1}^s (1 - y_i)^{-1}, \quad (3.22)$$

where the conservation laws have been used.

The quotient of determinants in (3.21) is the Schur polynomial associated with the partition λ^n . Actually it reduces to the elementary symmetric polynomial

$$\alpha_n^s(y_1, \dots, y_s) = \frac{1}{n!(s-n)!} \sum_{\sigma_s} \prod_{i=1}^n y_{\sigma_s(i)}, \quad (3.23)$$

since λ^n is the minimal partition of degree n . In the expression above the sum runs over all permutations of degree s and we have defined $\alpha_0^s = 1$. We thus may write

$$\begin{aligned} \mathcal{P}^{-1} \partial_1 \cdots \partial_s \mathcal{P} &= \left[\prod_{i=1}^s y_i^{-1} (1 - y_i)^{-1} \right] \frac{\Gamma(1 + j_1 - m_1)}{\Gamma(-j_3 + m_3)} \\ &\times \sum_{n=0}^s (-1)^{s+n} \frac{\Gamma(-j_3 + m_3 + n)}{\Gamma(1 - s + j_1 - m_1 + n)} \alpha_n^s(y_1, \dots, y_s), \end{aligned} \quad (3.24)$$

where we have used the conservation laws and, repeatedly, the relation

$$\Gamma(1 + z - n) = (-1)^n \frac{\Gamma(1 + z) \Gamma(-z)}{\Gamma(n - z)}, \quad (3.25)$$

which holds for $n \in \mathbb{N}$.

Therefore, the three-point function is given by

$$\begin{aligned} \mathcal{A}_3 &= \Gamma(-s) \frac{\Gamma(1 + j_1 - m_1)}{\Gamma(-j_3 + m_3)} \frac{\Gamma(1 + j_1 - \bar{m}_1)}{\Gamma(-j_3 + \bar{m}_3)} \sum_{n, \bar{n}=0}^s \frac{\Gamma(-j_3 + m_3 + n)}{\Gamma(1 - s + j_1 - m_1 + n)} \\ &\times \frac{\Gamma(-j_3 + \bar{m}_3 + \bar{n})}{\Gamma(1 - s + j_1 - \bar{m}_1 + \bar{n})} (-1)^{n+\bar{n}} \mathcal{A}_s^{n, \bar{n}}(-2j_1\rho, -2j_2\rho, \rho), \end{aligned} \quad (3.26)$$

where we have introduced the following integrals:

$$\begin{aligned} \mathcal{A}_s^{n, \bar{n}}(-2j_1\rho, -2j_2\rho, \rho) &= \int \prod_{i=1}^s d^2 y_i \prod_{i=1}^s |y_i|^{-4j_1\rho-2} |1 - y_i|^{-4j_2\rho-2} \\ &\times \prod_{i < j}^s |y_i - y_j|^{4\rho} \alpha_n^s(y_1, \dots, y_s) \alpha_{\bar{n}}^s(\bar{y}_1, \dots, \bar{y}_s). \end{aligned} \quad (3.27)$$

In the next subsection we will perform these multiple integrations.

3.2 Aomoto integrals in the complex plane

We need to calculate the following expression:

$$\mathcal{A}_s^{n,\bar{n}} = \int d^2y \prod_{i=1}^s |y_i|^{2a-2} |1-y_i|^{2b-2} \prod_{i<j}^s |y_i-y_j|^{4\rho} \alpha_n^s(y) \alpha_{\bar{n}}^s(\bar{y}), \quad (3.28)$$

where we are writing $\alpha_n^s(y)$ instead of $\alpha_n^s(y_1, \dots, y_s)$ and d^2y instead of $\prod_{l=1}^s d^2y_l$. We will do it by using the results found by Aomoto [21] and a contour manipulation similar to the one discussed in [23]. We present the details here and include several useful formulas in the Appendix.

Let us start transforming these complex integrals into multiple contour integrals following [29]. It is convenient to introduce the set of real variables $y_l = u_l + iv_l$ for $l = 1, 2, \dots, s$, in terms of which $\mathcal{A}_s^{n,\bar{n}}(a, b, \rho)$ takes the form

$$\begin{aligned} \mathcal{A}_s^{n,\bar{n}} &= \int du dv \prod_{l=1}^s (u_l^2 + v_l^2)^{a-1} ((1-u_l)^2 + v_l^2)^{b-1} \\ &\quad \times \prod_{l<m}^s ((u_l - u_m)^2 + (v_l - v_m)^2)^{2\rho} \alpha_n^s(u + iv) \alpha_{\bar{n}}^s(u - iv). \end{aligned} \quad (3.29)$$

The integrations are now performed on the real axis.

Next, we analytically continue the contours of integration of the v 's and we shift them close to the imaginary axis, i.e., we perform the change of variables: $v_l \rightarrow -i \exp(-2i\epsilon) v_l$, where ϵ is a vanishingly small positive number. Thus we may now rewrite the previous expression as

$$\begin{aligned} \mathcal{A}_s^{n,\bar{n}} &= (-i)^s \int du dv \prod_{l=1}^s (u_l^2 - e^{-4i\epsilon} v_l^2)^{a-1} ((1-u_l)^2 - e^{-4i\epsilon} v_l^2)^{b-1} \\ &\quad \times \prod_{l<m}^s ((u_l - u_m)^2 - e^{-4i\epsilon} (v_l - v_m)^2)^{2\rho} \alpha_n^s(u + e^{-2i\epsilon} v) \alpha_{\bar{n}}^s(u - e^{-2i\epsilon} v). \end{aligned} \quad (3.30)$$

An irrelevant phase factor of the form $e^{-2is\epsilon}$ was omitted in the previous equation.

The additional change of integration variables $z_l = u_l + v_l$ and $w_l = u_l - v_l$ gives the following form for the integral (3.28):

$$\begin{aligned} \mathcal{A}_s^{n,\bar{n}} &= \int dz dw \prod_{l=1}^s (z_l - i\epsilon(z_l - w_l))^{a-1} (w_l + i\epsilon(z_l - w_l))^{a-1} (1 - z_l + i\epsilon(z_l - w_l))^{b-1} \\ &\quad \times (1 - w_l - i\epsilon(z_l - w_l))^{b-1} \prod_{l<m}^s (z_l - z_m - i\epsilon(z_l - w_l + z_m - w_m))^{2\rho} \\ &\quad \times (w_l - w_m + i\epsilon(z_l - w_l + z_m - w_m))^{2\rho} \alpha_n^s(z - i\epsilon(z - w)) \alpha_{\bar{n}}^s(w + i\epsilon(z - w)), \end{aligned} \quad (3.31)$$

where we have written $dz dw$ in place of $\prod_{l=1}^s (-idz_l dw_l/2)$.

After performing the limit $\epsilon \rightarrow 0^+$ this last double integral factorizes as a product of two single contour integrals. The ϵ -dependent terms determine how the integration contours should be deformed in order to avoid the singularities at 0 and 1 and to keep them away from each other. The order in which the integrations in the z 's are to be made define the way in which the integration contours corresponding to

the w 's should be arranged: if $z_i < z_j$ then the contour of w_i must lie below the one of w_j . Then, the limit $\epsilon \rightarrow 0^+$ must be performed. See [23] for more details on this kind of manipulation of complex contours.

The integral (3.31) can therefore be written as

$$\mathcal{A}_s^{n,\bar{n}} = \left(-\frac{i}{2}\right)^s \sum_{\sigma} A_{\sigma}^n(a, b, \rho) \times J_{\sigma}^{\bar{n}}(a, b, \rho), \quad (3.32)$$

where σ runs over all orderings of the z 's. $A_{\sigma}^n(a, b, \rho)$ denotes the integrals over the z 's ordered according to σ and $J_{\sigma}^{\bar{n}}(a, b, \rho)$ denotes the contour integrals of the w 's with the prescription on the contours that follows, as we have described, from σ .

If one of the z 's is not in the interval $(0, 1)$, then at least one of the contours of the w 's can be deformed to infinity and thus the integral vanishes. On the other hand, since $\alpha_n^s(z_1, \dots, z_n)$ is a symmetric polynomial, the integration limits in $A_{\sigma}^n(a, b, \rho)$ can be freely set to 0 and 1, showing that actually $A_{\sigma}^n(a, b, \rho)$ does not depend on σ but only on s . It is given by the following integral:

$$A_s^n(a, b, \rho) = \int_0^1 dz_1 \cdots \int_0^1 dz_s \prod_{i=1}^s z_i^{a-1} (1 - z_i)^{b-1} \prod_{i < j}^s (z_i - z_j)^{2\rho} \alpha_n^s(z). \quad (3.33)$$

This is Aomoto integral of order n whose explicit expression we recall in the Appendix.

The integral $J_{\sigma}^{\bar{n}}(a, b, \rho)$ is given by

$$J_{\sigma}^{\bar{n}}(a, b, \rho) = \int_{\Lambda_1} dw_1 \cdots \int_{\Lambda_s} dw_s \prod_{i=1}^s w_i^{a-1} (1 - w_i)^{b-1} \prod_{i < j}^s (w_i - w_j)^{2\rho} \alpha_{\bar{n}}^s(w), \quad (3.34)$$

where every contour Λ_i comes from $-\infty$ in the lower half complex plane and goes to $+\infty$ in the upper half complex plane crossing the real line in $(0, 1)$ with none of the contours intersecting another one. Notice that all these contours can be deformed in such a way that each integration can be performed from $+\infty$ in the lower half complex plane to $+\infty$ in the upper half complex plane encircling the singularity at 1 clockwise and the contours do not intersect with each other.

The factor $\prod_{i < j}^s (w_i - w_j)^{2\rho}$ is defined so that if all the w_i are placed on the real axis and decreasingly ordered, then the phases of the multivalued products are all equal to zero. When the variable w_i is continued along its contour and is taken around some other point, say w_j , in such a way that the contour of w_i goes above w_j , the product $\prod_{i < j}^s (w_i - w_j)^{2\rho}$ gets an additional phase factor $\exp(-2\pi i \rho)$. It follows that the integral (3.34) can be identified, up to phase factors depending neither on n nor on \bar{n} , with

$$J_s^{\bar{n}}(a, b, \rho) = \int_1^{\infty} dw_1 \cdots \int_1^{\infty} dw_s \prod_{i=1}^s w_i^{a-1} (1 - w_i)^{b-1} \prod_{i < j}^s |w_i - w_j|^{2\rho} \alpha_{\bar{n}}^s(w), \quad (3.35)$$

which is, again, independent of σ .

Performing the change of variables $w_i \rightarrow y_i = 1/w_i$ in (3.35) and using the inversion identity

$$\alpha_{s-\bar{n}}^s(1/y_1, \dots, 1/y_s) = \left[\prod_{i=1}^s y_i^{-1} \right] \alpha_{\bar{n}}^s(y_1, \dots, y_s), \quad (3.36)$$

one gets that $J_s^{\bar{n}}(a, b, \rho)$ equals $A_s^{s-\bar{n}}(-a-b-2\rho(s-1), b, \rho)$.

Summarizing, we have found that

$$\mathcal{A}_s^{n,\bar{n}} = C A_s^n(a, b, \rho) A_s^{s-\bar{n}}(-a - b - 2\rho(s-1), b, \rho), \quad (3.37)$$

where the factor C depends neither on n nor on \bar{n} . Actually, the independence of this factor on these parameters allows us to read it from Dotsenko-Fateev integral (formula (B.9) of [23]) that corresponds to the case $n = \bar{n} = 0$.

We finally get

$$\mathcal{A}_s^{n,\bar{n}} = \binom{s}{n} \binom{s}{\bar{n}} \frac{\Gamma(\alpha + s)\Gamma(2 - \alpha - \beta - 2s)}{\Gamma(1 - \alpha - s)\Gamma(\alpha + \beta + 2s - 1)} \frac{\Gamma(\alpha + \beta + 2s - n - 1)\Gamma(1 - \alpha - s + \bar{n})}{\Gamma(\alpha + s - n)\Gamma(2 - \alpha - \beta - 2s + \bar{n})} \times \mathcal{S}_s, \quad (3.38)$$

where $\alpha = a/\rho$, $\beta = b/\rho$ and \mathcal{S}_s is Dotsenko-Fateev integral.

Similarly as in the real case, all Aomoto integrals of definite order in the complex plane can be placed together in a single expression (see the Appendix). In fact, let us consider the integral

$$\mathcal{A}_s(z, \bar{z}) = \int d^2y \prod_{i=1}^s |y_i|^{2a-2} |1 - y_i|^{2b-2} |z - y_i|^2 \prod_{i < j}^s |y_i - y_j|^{4\rho}. \quad (3.39)$$

This is a polynomial in z and \bar{z} whose coefficients precisely are, up to a phase, those integrals that we have evaluated. Replacing their expressions into (3.39) we get

$$\mathcal{A}_s(z, \bar{z}) = \frac{1}{4^s} |\bar{P}_s^{\alpha-1, \beta-1}(1-2z)|^2 \mathcal{S}_s, \quad (3.40)$$

where $\bar{P}_s^{\alpha, \beta}$ are the monic Jacobi polynomials whose definition we recall in equation (6.11) in the Appendix. Equivalently we may write

$$\mathcal{A}_s(z, \bar{z}) = \frac{\gamma(\alpha + s)}{\gamma(\alpha)} \frac{\gamma(\alpha + \beta + s - 1)}{\gamma(\alpha + \beta + 2s - 1)} |{}_2F_1(-s, \alpha + \beta + s - 1; \alpha; z)|^2 \mathcal{S}_s. \quad (3.41)$$

We may now go back to (3.26) and perform the sums.

3.3 Analytic continuation

Let us recall that we have to evaluate the following sums:

$$\begin{aligned} \mathcal{A}_3 &= \Gamma(-s) \frac{\Gamma(1 + j_1 - m_1)}{\Gamma(-j_3 + m_3)} \frac{\Gamma(1 + j_1 - \bar{m}_1)}{\Gamma(-j_3 + \bar{m}_3)} \sum_{n, \bar{n}=0}^s \frac{\Gamma(-j_3 + m_3 + n)}{\Gamma(1 - s + j_1 - m_1 + n)} \\ &\quad \times \frac{\Gamma(-j_3 + \bar{m}_3 + \bar{n})}{\Gamma(1 - s + j_1 - \bar{m}_1 + \bar{n})} (-1)^{n+\bar{n}} \mathcal{A}_s^{n,\bar{n}}(-2j_1\rho, -2j_2\rho, \rho), \end{aligned} \quad (3.42)$$

where $\mathcal{A}_s^{n,\bar{n}}(-2j_1\rho, -2j_2\rho, \rho)$ is given in (3.38) with the replacements $a = -2j_1\rho$ and $b = -2j_2\rho$.

First note that the combinatorial numbers in (3.38) allow us to freely extend the sum to ∞ and write the result in terms of a generalized hypergeometric function. In fact, this leads to the result found in reference [15], namely,

$$\mathcal{A}_{m_1 m_2 m_3}^{j_1 j_2 j_3} = \mathcal{C}\bar{\mathcal{C}}\mathcal{I}(j_1, j_2, j_3, \rho), \quad (3.43)$$

where

$$\mathcal{C} = \frac{\Gamma(-2j_3)\Gamma(1+j_2+m_2)\Gamma(1+j_2+j_3+m_1)}{\Gamma(-j_3-m_3)\Gamma(-j_1+m_1)\Gamma(1-m_1-j_3+j_2)} {}_3F_2 \left[\begin{matrix} -j_3+m_3, -m_1-j_1, 1-m_1+j_1 \\ -m_1-j_2-j_3, 1-m_1+j_2-j_3 \end{matrix} \middle| 1 \right]. \quad (3.44)$$

$\overline{\mathcal{C}}$ is the same expression with the replacement $m_i \rightarrow \overline{m}_i$ and $\mathcal{I}(j_1, j_2, j_3, \rho) = \Gamma(-s)\mathcal{S}_s(-2j_1\rho, -2j_2\rho, \rho)$.

It was pointed out by Y. Satoh [20] that this result agrees, up to a phase, with the integral transform to the m -basis of the three-point function computed by J. Teschner [5] in the x -basis, only when the amplitude involves at least one state from the discrete representation. This seems to be a natural conclusion of the procedure implemented in reference [15], where the starting point is a three-point function containing one highest-weight state, and then acting with the lowering operator J_0^- and using the Baker-Campbell-Hausdorff formula, the dependence on $m_1 = j_1$ is changed in an integer amount. Indeed, the sums leading to (3.43) in reference [15] sweep the highest-weight representation. But we are considering three generic states here and we arrive at the same result. The common assumption in both procedures though is that the number of screening operators s is an integer number. The analytic continuation to non-integer values of s was performed in [15] for the particular case of on-shell tachyons in the $SL(2, \mathbb{R})/U(1)$ coset model representing the two dimensional black hole. We now use monodromy invariance to analytically continue the dependence on s and then show that this leads to the complete exact result for generic three-point functions.

Let us start by noticing that equation (3.38) may be recast as follows:

$$\mathcal{A}_s^{n, \overline{n}}(-2j_1\rho, -2j_2\rho, \rho) = \frac{(-1)^{n+\overline{n}}}{\pi\Gamma(0)} \frac{\gamma(s-2j_1-2j_2-1)}{\gamma(2s-2j_1-2j_2-1)} \frac{\gamma(-2j_1+s)}{\gamma(-2j_1)} \mathcal{S}_s(-2j_1\rho, -2j_2\rho, \rho) I(n, \overline{n}, s), \quad (3.45)$$

where

$$I(n, \overline{n}, s) = \frac{\pi\gamma(-2j_1)}{\gamma(-s)\gamma(-j_{12})} \frac{\Gamma(n-s)\Gamma(-\overline{n})\Gamma(1+2j_3-\overline{n})\Gamma(-j_{23}+n)}{\Gamma(1+s+\overline{n})\Gamma(1+n)\Gamma(-2j_3+n)\Gamma(1+j_{23}-\overline{n})}. \quad (3.46)$$

This combination of Γ -functions can be rewritten using the formula derived in reference [4], namely,

$$I(n, \overline{n}, s) = \int d^2u u^{-s+n-1} \overline{u}^{-s+\overline{n}-1} (|F(-s, s-2j_1-2j_2-1, -2j_1; u)|^2 + \lambda |u^{1+2j_1} F(s-2j_2, 1-s+2j_1, 2+2j_1; u)|^2), \quad (3.47)$$

with

$$\lambda = -\frac{\gamma(-2j_1)^2\gamma(1-s+2j_1)\gamma(s-2j_2)}{(1+2j_1)^2\gamma(-s)\gamma(s-2j_1-2j_2-1)}.$$

Here F denotes the hypergeometric function ${}_2F_1$.

The factor $\gamma(-s)$ in the denominator of λ diverges for integer s . Therefore the second term in the integral (3.47) would not contribute in this case. However, as discussed in [4], the sum of hypergeometric functions in (3.47) is the unique monodromy invariant combination. So we claim that the full expression has to be used in order to properly analytically continue s to non-integer values.

The requirement of monodromy invariance could seem unnatural in the context of three-point functions since the simplest non-trivial case among multipoint correlators that involves invariance under the action of the monodromy group is the four-point function. A way to understand this follows from the fact that any three-point function can be obtained by acting on a particular well defined four-point-like function with a suitable differential operator. Indeed, from (3.26) and (3.41) it follows that

$$\mathcal{A}_3 = \Gamma(-s)\mathcal{O}\overline{\mathcal{O}}\mathcal{A}_s(z, \overline{z})|_{z, \overline{z}=0}, \quad (3.48)$$

where

$$\mathcal{O} = \frac{\Gamma(1+j_1-m_1)}{\Gamma(-j_3+m_3)\Gamma(-s)\Gamma(1+s)} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(-j_3+m_3+n)\Gamma(-s+n)}{\Gamma(1-s+j_1-m_1+n)} \partial^{s-n}. \quad (3.49)$$

The function $\mathcal{A}_s(z, \bar{z})$ is explicitly given in (3.41) and it is related to the four-point function corresponding to two highest-weight states with spins $j_1 - \frac{k}{2} + 1$ and $j = \frac{k}{2} - 1$ at the points 0 and z , respectively. We now show that monodromy invariant analytic continuation to non-integer values of s of equation (3.41) leads to the exact result quoted in (2.49).

In order to obtain the explicit expression for the three-point function we will make use of the integral representation of the integrand in (3.47) given in [4], namely,

$$\begin{aligned} |F(a, b; c; u)|^2 + \lambda |u^{1-c} F(1+b-c, 1+a-c; 2-c; u)|^2 \\ = \frac{\gamma(c)}{\pi \gamma(b) \gamma(c-b)} |u^{1-c}|^2 \int d^2 t |t|^{b-1} (u-t)^{c-b-1} (1-t)^{-a}|^2, \end{aligned} \quad (3.50)$$

which allows to write the three-point function as

$$\begin{aligned} \mathcal{A}_3 &= \frac{\Gamma(-s)}{\pi^2 \Gamma(0)} \mathcal{S}_s(-2j_1\rho, -2j_2\rho, \rho) \frac{\gamma(-2j_1+s)\gamma(-2j_2+s)}{\gamma(2s-2j_1-2j_2-1)} \frac{\Gamma(1+j_1-m_1)}{\Gamma(-j_3+m_3)} \\ &\times \frac{\Gamma(1+j_1-\bar{m}_1)}{\Gamma(-j_3+\bar{m}_3)} \sum_{n, \bar{n}=0}^{+\infty} \frac{\Gamma(-j_3+m_3+n)}{\Gamma(1-s+j_1-m_1+n)} \frac{\Gamma(-j_3+\bar{m}_3+\bar{n})}{\Gamma(1-s+j_1-\bar{m}_1+\bar{n})} \\ &\times \int d^2 u d^2 t u^n \bar{u}^{\bar{n}} |u|^{-2s+4j_1} |t|^{2s-4j_1-4j_2-4} |u-t|^{-2s+4j_2} |1-t|^{2s}. \end{aligned} \quad (3.51)$$

Then notice that the sum in n can be written in terms of yet another hypergeometric function as

$$\sum_{n=0}^{+\infty} \frac{\Gamma(-j_3+m_3+n)}{\Gamma(1-s+j_1-m_1+n)} u^n = \frac{\Gamma(-j_3+m_3)}{\Gamma(1-s+j_1-m_1)} F(-j_3+m_3, 1, 1-s+j_1-m_1; u). \quad (3.52)$$

Adding the antiholomorphic dependence and completing again the monodromy invariant combination we may use the integral representation (3.50) and write the three-point function in terms of the following integral:

$$\begin{aligned} \int d^2 u d^2 t d^2 z u^{j_1+m_1} \bar{u}^{j_1+\bar{m}_1} (u-z)^{-1-j_2+m_2} (\bar{u}-\bar{z})^{-1-j_2+\bar{m}_2} \\ \times z^{-1-j_3+m_3} \bar{z}^{-1-j_3+\bar{m}_3} |t|^{-2-2j_{12}} |t-u|^{-2-2j_{13}} |1-t|^{2s} |1-z|^{-2}. \end{aligned} \quad (3.53)$$

In order to solve this triple integral it is convenient to perform the change of variables $u \rightarrow u/z$, $t \rightarrow t/z$ and integrate z . Using the identity $F(a, b; c; t) = (1-t)^{c-a-b} F(c-a, c-b; c; t)$ and recalling that $F(0, b; c; t) = F(a, 0; c; t) = 1$, the three-point function takes the following form:

$$\begin{aligned} \mathcal{A}_3 &= \frac{2\Gamma(-s)}{\pi^2} \frac{\gamma(-2j_1+s)\gamma(-2j_2+s)}{\gamma(2s-2j_1-2j_2-1)} \frac{\Gamma(1+j_1-m_1)}{\Gamma(-j_1+\bar{m}_1)} \frac{\Gamma(1+j_2-\bar{m}_2)}{\Gamma(-j_2+m_2)} \frac{\Gamma(1+j_3-\bar{m}_3)}{\Gamma(-j_3+\bar{m}_3)} \\ &\times \mathcal{S}_s(-2j_1\rho, -2j_2\rho, \rho) \int d^2 u d^2 t u^{j_1+m_1} \bar{u}^{j_1+\bar{m}_1} (u-1)^{-1-j_2+m_2} (\bar{u}-1)^{-1-j_2+\bar{m}_2} \\ &\times |t|^{2(-1-j_{12})} |t-u|^{2(-1-j_{13})} |1-t|^{2s}. \end{aligned} \quad (3.54)$$

Finally this double integral can be carried out using the formula derived in [28], which we have collected in equation (6.12) in the Appendix. The final result for the three-point function, after performing Dotsenko-Fateev integral \mathcal{S}_s (see (6.2) in the Appendix) is the following:

$$\mathcal{A}_3 = -\pi^3 \frac{C(-j_1, -j_2, 1+j_3)W(j_1, -1-j_2, -1-j_3; m_i, \bar{m}_i)B(1+j_3)}{c_{m_1, \bar{m}_1}^{j_1} c_{m_2, \bar{m}_2}^{j_2} \prod_i (1+2j_i)}, \quad (3.55)$$

where the function $W(j_i, m_i, \bar{m}_i)$ appears in [20] and we recall it here for completeness

$$W(j_i, m_i, \bar{m}_i) = \left(\frac{i}{2}\right)^2 [C^{12} \bar{P}^{12} + C^{21} \bar{P}^{21}], \quad (3.56)$$

with

$$\begin{aligned} \left(\frac{i}{2}\right)^2 P^{12} &= s(j_1 - m_1)s(j_2 - m_2)C^{31} - s(j_2 - m_2)s(j_3 - j_2 - m_1)C^{13}, \\ C^{12} &= \frac{\Gamma(-1-j_1-j_2-j_3)\Gamma(1+j_3+m_3)}{\Gamma(-j_1+m_3)} G \left[\begin{matrix} 1+j_2-m_2, & -j_{13}, & -j_3+m_3 \\ m_3-j_1+j_2+1, & -m_2-j_1-j_3 \end{matrix} \right], \\ C^{31} &= \frac{\Gamma(1+j_3-m_3)\Gamma(1+j_3+m_3)}{\Gamma(1+j_1+j_2+j_3)} G \left[\begin{matrix} 1+j_2+m_2, & 1+j_1+j_2+j_3, & 1+j_1-m_1 \\ 2-m_1+j_2+j_3, & 2+m_2+j_1-j_3 \end{matrix} \right], \end{aligned} \quad (3.57)$$

and C^{21}, C^{13}, P^{21} are obtained exchanging j_1, m_1 and j_2, m_2 in C^{12}, C^{31} and P^{12} , respectively. G is defined by

$$G \left[\begin{matrix} a, & b, & c \\ e, & f \end{matrix} \right] = \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(e)\Gamma(f)} {}_3F_2 \left[\begin{matrix} a, & b, & c \\ e, & f \end{matrix} \middle| 1 \right]. \quad (3.58)$$

In order to show that our result (3.55) agrees with the expression quoted in [20], namely Eq. (2.49), it is convenient to recall the relation (2.34) between $V_{j,m,\bar{m}}$ and $\Phi_{j,m,\bar{m}}$ and use the following identity derived in [20]

$$c_{m_3, \bar{m}_3}^{j_3} W(j_i; m_i, \bar{m}_i) C(1+j_i) = B(j_3) W(j_1, j_2, -1-j_3; m_i, \bar{m}_i) C(1+j_1, 1+j_2, -j_3). \quad (3.59)$$

Similarly as in [20] the agreement is up to a phase.

This completes the proof that the spectral flow conserving three-point function (3.55) computed using the Coulomb gas formalism is indeed the integral transform to the m -basis of the expression computed in [5].

4 Spectral flow non-conserving three-point functions

In this section we compute the spectral flow non-conserving three-point function in the Coulomb gas approach. We first implement the prescription introduced in [6, 4] and then we propose a more natural novel procedure.

The method discussed in [6] for computing spectral flow non-conserving n -point functions starts with the evaluation of a correlator in the x -basis involving n physical states in the $w=0$ sector and one spectral flow operator, namely, $\Phi_{\frac{k}{2}-1}$, for every unit of winding number to be violated⁹. This correlation function

⁹Notice that, up to a k -dependent factor, we can freely take $\Phi_{\frac{k}{2}-1}$ as the spectral flow operator instead of $\Phi_{-\frac{k}{2}}$, which was the choice made in [4], since this is its $j \leftrightarrow (-1-j)$ reflected vertex.

is transformed to the m -basis assuming that for the spectral flow operator m equals $\pm\frac{k}{2}$, depending on the sign of winding number it transfers. Fourier integrals involving spectral flow operators usually diverge for these values of m . This problem is overcome choosing the normalization $V_{conf}^{-1}\Phi_{\frac{k}{2}-1}$, where V_{conf} is the volume of the conformal group of S^2 with two points fixed [4]. The spectral flow operation is then completed dividing the correlation function by

$$\prod_{i<j}(\zeta_i - \zeta_j)^{\pm\frac{k}{2}} \prod_{i,a}(\zeta_i - z_a)^{\pm m_a}, \quad (4.1)$$

where ζ_i and z_a are the points in which the spectral flow operators and the physical vertices are inserted, respectively, and again the signs of the exponents depend on the sign of the m of each of the spectral flow operators that were introduced. After this operation, the dependence of the result on the ζ_i should disappear and the final expression is identified with a winding number non-conserving correlation function.

Among the properties of the operator $\Phi_{\frac{k}{2}-1}$, three are of particular relevance for deriving the previous procedure. Let us briefly recall them here. First, $\Phi_{\frac{k}{2}-1}$ is a degenerate field with null descendants $J_{-1}^{\pm}\Phi_{\frac{k}{2}-1}$ (according to the sign of m) and so every correlator containing it verifies an easy solvable KZ equation [6]. Second, $\Phi_{\frac{k}{2}-1}$ satisfies the following fusion rule $\Phi_{\frac{k}{2}-1}\Phi_j \sim \Phi_{\frac{k}{2}-2-j}$. Finally, the conformal dimension of $\Phi_{\frac{k}{2}-1}$ is $-\frac{k}{4}$. Then, if this prescription is implemented in the m -basis, the parafermionic content of $\Phi_{\frac{k}{2}-1}$ with $m = \pm\frac{k}{2}$ has vanishing dimension and so it can be identified with the identity. These last two properties allow to interpret $\Phi_{\frac{k}{2}-1}$ as a spectral flow operator transferring one unit of winding number.

This prescription was used in [4] in order to explicitly obtain the one unit spectral flow three-point function. Now we will reproduce this computation using free fields. In order to implement this procedure in the Coulomb gas approach we need to translate all computations from the x -basis to the m -basis. So, we need to evaluate the coefficient of the term which goes like $\zeta^{m_1}\bar{\zeta}^{\bar{m}_1}$ when $\zeta, \bar{\zeta} \rightarrow 0$ in the following four-point function:

$$\mathcal{A}_4 = \frac{\Gamma(-s)}{V_{conf}} \left\langle \tilde{V}_{j_1, m_1, \bar{m}_1}(0) \tilde{V}_{\frac{k}{2}-1, \frac{k}{2}, \frac{k}{2}}(\zeta) \tilde{V}_{j_2, m_2, \bar{m}_2}(1) \tilde{V}_{j_3, m_3, \bar{m}_3}(+\infty) \mathcal{S}^s \right\rangle, \quad (4.2)$$

where the conservation laws obtained from the zero mode integration of the fields are

$$s = \frac{k}{2} + j_1 + j_2 + j_3, \quad (4.3)$$

$$\frac{k}{2} + m_1 + m_2 + m_3 = 0, \quad (4.4)$$

$$\frac{k}{2} + \bar{m}_1 + \bar{m}_2 + \bar{m}_3 = 0, \quad (4.5)$$

and all the vertex operators create $w = 0$ states.

Performing all the contractions in (4.2) we obtain

$$\begin{aligned} \mathcal{A}_4 = & \frac{\Gamma(-s)}{V_{conf}} |\zeta|^{-2j_1} |\zeta - 1|^{-2j_2} \int [dy] \prod_{i=1}^s |y_i|^{-4j_1\rho} |1 - y_i|^{-4j_2\rho} |\zeta - y_i|^2 \\ & \times \prod_{i<j}^s |y_i - y_j|^{4\rho} \mathcal{Q}^{-1} \partial_1 \cdots \partial_s \mathcal{Q} \times \bar{\mathcal{Q}}^{-1} \bar{\partial}_1 \cdots \bar{\partial}_s \bar{\mathcal{Q}}, \end{aligned} \quad (4.6)$$

where now the $u - v$ contribution is determined by

$$\mathcal{Q} = \det \left[(\zeta - y_i)(1 - y_i)^{-(j_2 - m_2)} y_i^{j-1-j_1+m_1} \right], \quad (4.7)$$

i.e., we have $\mathcal{Q} = \Lambda \mathcal{P}$, with

$$\Lambda = \prod_{i=1}^s (\zeta - y_i). \quad (4.8)$$

In order to compute (4.6), we first discuss the evaluation of the ghost contribution and then perform the multiple integrations. We will consider in the next subsection the case in which one of the vertices, say $V_{j_1, m_1, \overline{m}_1}$, creates a highest-weight state, i.e., we will assume that $m_1 = \overline{m}_1 = j_1$ and then we will relax this assumption.

4.1 Evaluation of the ghost correlator

The derivatives of \mathcal{Q} appearing in (4.6) when $m_1 = \overline{m}_1 = j_1$ can be written as

$$\partial_1 \cdots \partial_s \mathcal{Q} = \det \left\{ \partial_i \left[(\zeta - y_i)(1 - y_i)^{-j_2+m_2} y_i^{j-1} \right] \right\} = y_i^{-1} (1 - y_i)^{-j_2+m_2-1} \sum_{l=0}^2 \ell_l^j y_i^{j-1+l}, \quad (4.9)$$

where

$$\ell_0^j = (j-1)\zeta, \quad (4.10)$$

$$\ell_1^j = -j + (1 - j + j_2 - m_2)\zeta, \quad (4.11)$$

$$\ell_2^j = (j - j_2 + m_2). \quad (4.12)$$

Therefore,

$$\mathcal{Q}^{-1} \partial_1 \cdots \partial_s \mathcal{Q} = \left[\prod_{i=1}^s y_i^{-1} (\zeta - y_i)^{-1} (1 - y_i)^{-1} \right] \frac{\det \left[\sum_{l=0}^2 \ell_l^j y_i^{j-1+l} \right]}{\det \left(y_i^{j-1} \right)}. \quad (4.13)$$

We can rewrite the quotient of determinants in (4.13) as

$$\frac{\det \left[\sum_{l=0}^2 \ell_l^j y_i^{j-1+l} \right]}{\det \left(y_i^{j-1} \right)} = \sum_{\mu} \left[\prod_{j=1}^s \ell_{\mu_{s+1-j}}^j \right] \frac{\det(y_i^{j-1+\mu_{s+1-j}})}{\det \left(y_i^{j-1} \right)}, \quad (4.14)$$

which formally seems to be a linear combination of Schur polynomials except that μ is not in general a partition but a s -uple with entries 0, 1 and 2. However, using that the permutation of two columns changes the sign of a determinant, it can be shown that only partitions contribute to this sum. Indeed s -uples of the form $(\dots, 0, 1, \dots)$ or $(\dots, 1, 2, \dots)$ are forbidden, and similarly those of the form $(\dots, 0, 2, 2, \dots)$ or $(\dots, 0, 0, 2, \dots)$ cannot occur. s -uples ending with 0 are also forbidden. Only s -uples of the form $\mu = (2, \dots, 2, 1, \dots, 1, 0, 2, 1, \dots, 1, 0, 2, 1, \dots, 1)$ contribute and they add up to the Schur polynomial

$s_\lambda(y_1, \dots, y_s)$ associated to the partition $\lambda = (2, \dots, 2, 1, \dots, 1)$ up to a $(-1)^t$ factor, where t is the number of times a couple "0, 2" was replaced by "1, 1" in μ in order to get λ .

Therefore, denoting the number of 2's in these partitions by n we can write

$$\frac{\det \left[\sum_{l=0}^2 \ell_l^j y_i^{j-1+l} \right]}{\det \left(y_i^{j-1} \right)} = \left[\prod_{i=1}^s y_i \right] \sum_{n=0}^s C_n \alpha_n^s(y_1, \dots, y_s), \quad (4.15)$$

where the C_n depend on ζ and we have used that

$$s_\lambda(y_1, \dots, y_s) = \left[\prod_{i=1}^s y_i \right] \times \alpha_n^s(y_1, \dots, y_s). \quad (4.16)$$

The contribution of the $u - v$ fields can be written as

$$\mathcal{Q}^{-1} \partial_1 \dots \partial_s \mathcal{Q} = \left[\prod_{i=1}^s (\zeta - y_i)^{-1} (1 - y_i)^{-1} \right] \sum_{n=0}^s C_n \alpha_n^s(y_1, \dots, y_s). \quad (4.17)$$

It can be shown by induction on n that C_n is given by

$$C_n = \frac{\Gamma(1 + s - j_2 + m_2)}{\Gamma(1 + s - n - j_2 + m_2)} \det A, \quad (4.18)$$

where

$$A = \begin{pmatrix} \ell_1^1 & \ell_0^2 & 0 & 0 & 0 & \dots & 0 \\ \ell_2^1 & \ell_1^2 & \ell_0^3 & 0 & 0 & \dots & 0 \\ 0 & \ell_2^2 & \ell_1^3 & \ell_0^4 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \ell_2^{s-n-2} & \ell_1^{s-n-1} & \ell_0^{s-n} \\ 0 & \dots & 0 & 0 & 0 & \ell_2^{s-n-1} & \ell_1^{s-n} \end{pmatrix}. \quad (4.19)$$

Let us denote the determinant of the submatrix formed by the first r rows and r columns of A by d_r . Notice that $\det A = d_{s-n}$. d_r is a polynomial in ζ of degree r and it is determined by the following recursion:

$$d_r = \ell_1^r d_{r-1} - \ell_2^{r-1} \ell_0^r d_{r-2}, \quad (4.20)$$

where $d_1 = \ell_1^1$ and $d_2 = \ell_1^1 \ell_1^2 - \ell_2^1 \ell_0^2$. Solving (4.20) we get

$$d_r = \frac{\Gamma(0)}{\Gamma(-r)} (1 - \zeta)^{j_2 - m_2}, \quad (4.21)$$

and therefore,

$$C_n = \frac{\Gamma(1 + s - j_2 + m_2)}{\Gamma(1 + s - n - j_2 + m_2)} \frac{\Gamma(0)}{\Gamma(-s + n)} (1 - \zeta)^{j_2 - m_2}. \quad (4.22)$$

Note that the factor $(1 - \zeta)^{j_2 - m_2}$ is exactly what one needs in order to convert the factor $(\zeta - 1)^{-j_2}$ in (4.6) into $(\zeta - 1)^{-m_2}$, as expected from the prescription in [6]. Let us stress here that it was necessary to evaluate the $u - v$ contribution in order to get the ζ -dependence, unlike in [6], where this dependence is deduced by looking at the $SL(2, \mathbb{R})/U(1)$ coset model.

Putting everything together, the ghost contribution amounts to

$$\begin{aligned} \mathcal{Q}^{-1} \partial_1 \cdots \partial_s \mathcal{Q} &= \left[\prod_{i=1}^s (\zeta - y_i)^{-1} (1 - y_i)^{-1} \right] \Gamma(0) (1 - \zeta)^{j_2 - m_2} \\ &\quad \times \sum_{n=0}^s \frac{\Gamma(1 + s - j_2 + m_2)}{\Gamma(1 + s - n - j_2 + m_2) \Gamma(-s + n)} \alpha_n^s(y_1, \dots, y_s), \end{aligned} \quad (4.23)$$

times a similar contribution from the antiholomorphic part. Recall that this expression holds only for $m_1 = \bar{m}_1 = j_1$.

Notice that the factors $(\zeta - y_i)^{-1}$ exactly cancel their inverse ones appearing in (4.6) and this allows us to write the four-point function as an Aomoto integral.

4.2 Evaluation of the three-point function

Replacing the result (4.23) for the ghost correlator into (4.6) we get

$$\begin{aligned} \mathcal{A}_4 &= \frac{\Gamma(-s)}{V_{conf}} \zeta^{-j_1} \bar{\zeta}^{-j_1} (\zeta - 1)^{-m_2} (\bar{\zeta} - 1)^{-\bar{m}_2} \frac{\Gamma(0)}{\Gamma(-j_3 + m_3)} \frac{\Gamma(0)}{\Gamma(-j_3 + \bar{m}_3)} \\ &\quad \times \sum_{n, \bar{n}}^s (-1)^{n+\bar{n}} \frac{\Gamma(-j_3 + m_3 + n)}{\Gamma(-s + n)} \frac{\Gamma(-j_3 + \bar{m}_3 + \bar{n})}{\Gamma(-s + \bar{n})} \mathcal{A}_s^{n, \bar{n}}(-2j_1\rho + 1, -2j_2\rho, \rho), \end{aligned} \quad (4.24)$$

where we have used the conservation laws (4.3). Notice that the dependence on ζ and $\bar{\zeta}$ is exactly as expected from the prescription in [6]. Therefore we get the following result for the four-point function involving one highest-weight state and one spectral flow operator:

$$\begin{aligned} \mathcal{A}_4 &= \frac{1}{\pi} \frac{\Gamma(-s)}{\Gamma(-j_3 + m_3)} \frac{\Gamma(0)}{\Gamma(-j_3 + \bar{m}_3)} \zeta^{-j_1} \bar{\zeta}^{-j_1} (\zeta - 1)^{-m_2} (\bar{\zeta} - 1)^{-\bar{m}_2} \\ &\quad \times \sum_{n, \bar{n}}^s (-1)^{n+\bar{n}} \frac{\Gamma(-j_3 + m_3 + n)}{\Gamma(-s + n)} \frac{\Gamma(-j_3 + \bar{m}_3 + \bar{n})}{\Gamma(-s + \bar{n})} \mathcal{A}_s^{n, \bar{n}}(-2j_1\rho + 1, -2j_2\rho, \rho). \end{aligned} \quad (4.25)$$

The factors $\Gamma(n - s)\Gamma(\bar{n} - s)$ in the denominator above allow to extend the sums to ∞ . Replacing the expression for $\mathcal{A}_s^{n, \bar{n}}(-2j_1\rho + 1, -2j_2\rho, \rho)$ that we have already found in terms of Dotsenko-Fateev integral we get, up to a k -dependent factor,

$$\mathcal{A}_4 = \frac{\Gamma(0)}{\pi \Gamma(-s)} \mathcal{S}_s(-2j_1\rho + 1, -2j_2\rho, \rho) \left[\zeta^{-j_1} (\zeta - 1)^{-m_2} F(-j_3 + m_3, \frac{k}{2} + j_1 - j_2 - j_3 - 1; -2j_3|1) \times \text{c.c.} \right].$$

Using

$$F(-j_3 + m_3, \frac{k}{2} + j_1 - j_2 - j_3 - 1; -2j_3|1) = \frac{\Gamma(-2j_3)\Gamma(1 - \frac{k}{2} - j_1 + j_2 - m_3)}{\Gamma(1 - \frac{k}{2} - j_1 + j_2 - j_3)\Gamma(-j_3 - m_3)}, \quad (4.26)$$

we find

$$\begin{aligned} \mathcal{A}_4 &= \frac{\Gamma(0)}{\pi \Gamma(-s)} \frac{\mathcal{S}_s(-2j_1\rho + 1, -2j_2\rho, \rho)}{\gamma(1 + 2j_3)\gamma(1 - \frac{k}{2} - j_1 + j_2 - j_3)} \\ &\quad \frac{\Gamma(1 + j_2 + m_2)}{\Gamma(-j_2 - \bar{m}_2)} \frac{\Gamma(1 + j_3 + \bar{m}_3)}{\Gamma(-j_3 - m_3)} \zeta^{-j_1} \bar{\zeta}^{-j_1} (\zeta - 1)^{-m_2} (\bar{\zeta} - 1)^{-\bar{m}_2}. \end{aligned} \quad (4.27)$$

Rewriting the Selberg integral in terms of $\tilde{C}(-j_i)$ we get

$$\mathcal{A}_4 = \frac{\tilde{C}(-j_i)}{\gamma(-s)} \prod_{i=1}^3 \frac{\gamma((1+2j_i)\rho)}{\gamma(1+2j_i)} \frac{\Gamma(1+2j_1)}{\Gamma(-2j_1)} \frac{\Gamma(1+j_2+m_2)}{\Gamma(-j_2-\bar{m}_2)} \frac{\Gamma(1+j_3+\bar{m}_3)}{\Gamma(-j_3-m_3)} \zeta^{-j_1} \bar{\zeta}^{-j_1} (\zeta-1)^{-m_2} (\bar{\zeta}-1)^{-\bar{m}_2}. \quad (4.28)$$

Finally, using the following identity:

$$\frac{\tilde{C}(-j_i)}{\gamma(-j_1-j_2-j_3-\frac{k}{2})} = \frac{1}{\rho^3} \frac{\tilde{C}(1+j_i)}{\gamma(j_1+j_2+j_3+3-\frac{k}{2})} \prod_{i=1}^3 \frac{\gamma(1+2j_i)}{\gamma((1+2j_i)\rho)}, \quad (4.29)$$

we obtain, up to k -dependent factors,

$$\mathcal{A}_4 = \frac{\tilde{C}(1+j_i)}{\gamma(j_1+j_2+j_3+3-\frac{k}{2})} \frac{\Gamma(1+2j_1)}{\Gamma(-2j_1)} \frac{\Gamma(1+j_2+m_2)}{\Gamma(-j_2-\bar{m}_2)} \frac{\Gamma(1+j_3+\bar{m}_3)}{\Gamma(-j_3-m_3)} \zeta^{-j_1} \bar{\zeta}^{-j_1} (\zeta-1)^{-m_2} (\bar{\zeta}-1)^{-\bar{m}_2}, \quad (4.30)$$

in complete agreement with the result in [4].

Recall that the last expression holds only if the state with spin j_1 is a highest-weight state. In order to relax this condition we use the Campbell-Backer-Hausdorff identity, namely

$$e^{\alpha J_0^-} V_{j,m,\bar{m}}(z) e^{-\alpha J_0^-} = e^{\alpha[J_0^-, -]} V_{j,m,\bar{m}}(z) = \sum_{n=0}^{\infty} \frac{(-\alpha)^n}{n!} \frac{\Gamma(1+j+m)}{\Gamma(1+j+m-n)} V_{j,m-n,\bar{m}}(z). \quad (4.31)$$

A similar expression is obtained if the antiholomorphic component of the vertices is considered.

From the cyclic property of the trace and the commutation relation of the screening operators with J_0^- we get

$$\begin{aligned} & \left\langle e^{\alpha J_0^-} V_{j_1,j_1,j_1}(0) V_{\frac{k}{2}-1,\frac{k}{2},\frac{k}{2}}(\zeta) e^{-\alpha J_0^-} V_{j_2,m_2,\bar{m}_2}(1) V_{j_3,m_3,\bar{m}_3}(+\infty) \mathcal{S}^s \right\rangle \\ &= \left\langle V_{j_1,j_1,j_1}(0) V_{\frac{k}{2}-1,\frac{k}{2},\frac{k}{2}}(\zeta) e^{-\alpha J_0^-} V_{j_2,m_2,\bar{m}_2}(1) e^{\alpha J_0^-} e^{-\alpha J_0^-} V_{j_3,m_3,\bar{m}_3}(+\infty) e^{\alpha J_0^-} \mathcal{S}^s \right\rangle. \end{aligned}$$

Moreover, using the fact that J_0^- annihilates $V_{-\frac{k}{2},\frac{k}{2},\frac{k}{2}}(\zeta)$ we obtain

$$\begin{aligned} & \left\langle e^{\alpha J_0^-} V_{j_1,j_1,j_1}(0) e^{-\alpha J_0^-} V_{\frac{k}{2}-1,\frac{k}{2},\frac{k}{2}}(\zeta) V_{j_2,m_2,\bar{m}_2}(1) V_{j_3,m_3,\bar{m}_3}(+\infty) \mathcal{S}^s \right\rangle \\ &= \left\langle V_{j_1,j_1,j_1}(0) V_{\frac{k}{2}-1,\frac{k}{2},\frac{k}{2}}(\zeta) e^{-\alpha J_0^-} V_{j_2,m_2,\bar{m}_2}(1) e^{\alpha J_0^-} e^{-\alpha J_0^-} V_{j_3,m_3,\bar{m}_3}(+\infty) e^{\alpha J_0^-} \mathcal{S}^s \right\rangle. \end{aligned}$$

So the four point function satisfies the following identity:

$$\begin{aligned} & \sum_{n_1=0}^{\infty} \frac{(-\alpha)^{n_1}}{n_1!} \frac{\Gamma(1+2j_1)}{\Gamma(1+2j_1-n_1)} \left\langle V_{j_1,j_1-n_1,j_1}(0) V_{\frac{k}{2}-1,\frac{k}{2},\frac{k}{2}}(\zeta) V_{j_2,m_2,\bar{m}_2}(1) V_{j_3,m_3,\bar{m}_3}(+\infty) \mathcal{S}^s \right\rangle \\ &= \sum_{n_2,n_3=0}^{\infty} \frac{\alpha^{n_2+n_3}}{n_2!n_3!} \frac{\Gamma(1+j_2+m_2)}{\Gamma(1+j_2+m_2-n_2)} \frac{\Gamma(1+j_3+m_3)}{\Gamma(1+j_3+m_3-n_3)} \\ & \quad \times \left\langle V_{j_1,j_1,j_1}(0) V_{\frac{k}{2}-1,\frac{k}{2},\frac{k}{2}}(\zeta) V_{j_2,m_2-n_2,\bar{m}_2}(1) V_{j_3,m_3-n_3,\bar{m}_3}(+\infty) \mathcal{S}^s \right\rangle. \quad (4.32) \end{aligned}$$

Identifying powers on both sides of (4.32) we find

$$\begin{aligned}
& \left\langle V_{j_1, j_1 - n_1, j_1}(0) V_{\frac{k}{2} - 1, \frac{k}{2}, \frac{k}{2}}(\zeta) V_{j_2, m_2, \overline{m}_2}(1) V_{j_3, m_3, \overline{m}_3}(+\infty) \mathcal{S}^s \right\rangle \\
&= \sum_{p=0}^{n_1} (-1)^{n_1} \binom{n_1}{p} \frac{\Gamma(1 + 2j_1 - n_1)}{\Gamma(1 + 2j_1)} \frac{\Gamma(1 + j_2 + m_2)}{\Gamma(1 + j_2 + m_2 - p)} \frac{\Gamma(1 + j_3 + m_3)}{\Gamma(1 + j_3 + m_3 - n_1 + p)} \\
&\quad \times \left\langle V_{j_1, j_1, j_1}(0) V_{\frac{k}{2} - 1, \frac{k}{2}, \frac{k}{2}}(\zeta) V_{j_2, m_2 - p, \overline{m}_2}(1) V_{j_3, m_3 - n_1 + p, \overline{m}_3}(+\infty) \mathcal{S}^s \right\rangle, \quad (4.33)
\end{aligned}$$

i.e.,

$$\begin{aligned}
& \left\langle V_{j_1, m_1, j_1}(0) V_{\frac{k}{2} - 1, \frac{k}{2}, \frac{k}{2}}(\zeta) V_{j_2, m_2, \overline{m}_2}(1) V_{j_3, m_3, \overline{m}_3}(+\infty) \mathcal{S}^s \right\rangle \\
&= \sum_{p=0}^{\infty} (-1)^{j_1 - m_1} \frac{\Gamma(1 + j_1 + m_1) \Gamma(1 + j_3 + m_3)}{p! \Gamma(1 + 2j_1) \Gamma(-j_1 + m_1) \Gamma(-j_2 - m_2)} \frac{\Gamma(-j_2 - m_2 + p) \Gamma(-j_1 + m_1 + p)}{\Gamma(1 + j_3 + m_3 - j_1 + m_1 + p)} \\
&\quad \times \left\langle V_{j_1, j_1, j_1}(0) V_{\frac{k}{2} - 1, \frac{k}{2}, \frac{k}{2}}(\zeta) V_{j_2, m_2 - p, \overline{m}_2}(1) V_{j_3, m_3 - j_1 + m_1 + p, \overline{m}_3}(+\infty) \mathcal{S}^s \right\rangle. \quad (4.34)
\end{aligned}$$

Taking into account the antiholomorphic component we obtain

$$\begin{aligned}
& \left\langle V_{j_1, m_1, \overline{m}_1}(0) V_{\frac{k}{2} - 1, \frac{k}{2}, \frac{k}{2}}(\zeta) V_{j_2, m_2, \overline{m}_2}(1) V_{j_3, m_3, \overline{m}_3}(+\infty) \mathcal{S}^s \right\rangle \\
&= \sum_{p, \overline{p}=0}^{\infty} (-1)^{j_1 - m_1} (-1)^{j_1 - \overline{m}_1} \frac{\Gamma(1 + j_1 + m_1) \Gamma(1 + j_3 + m_3)}{p! \Gamma(1 + 2j_1) \Gamma(-j_1 + m_1) \Gamma(-j_2 - m_2)} \frac{\Gamma(-j_2 - m_2 + p) \Gamma(-j_1 + m_1 + p)}{\Gamma(1 + j_3 + m_3 - j_1 + m_1 + p)} \\
&\quad \times \frac{\Gamma(1 + j_1 + \overline{m}_1) \Gamma(1 + j_3 + \overline{m}_3)}{\overline{p}! \Gamma(1 + 2j_1) \Gamma(-j_1 + \overline{m}_1) \Gamma(-j_2 - \overline{m}_2)} \frac{\Gamma(-j_2 - \overline{m}_2 + \overline{p}) \Gamma(-j_1 + \overline{m}_1 + \overline{p})}{\Gamma(1 + j_3 + \overline{m}_3 - j_1 + \overline{m}_1 + \overline{p})} \\
&\quad \times \left\langle V_{j_1, j_1, j_1}(0) V_{\frac{k}{2} - 1, \frac{k}{2}, \frac{k}{2}}(\zeta) V_{j_2, m_2 - p, \overline{m}_2 - \overline{p}}(1) V_{j_3, m_3 - j_1 + m_1 + p, \overline{m}_3 - j_1 + \overline{m}_1 + \overline{p}}(+\infty) \mathcal{S}^s \right\rangle. \quad (4.35)
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathcal{A}_4 &= \frac{\tilde{C}(1 + j_i)}{\gamma(j_1 + j_2 + j_3 + 3 - \frac{k}{2})} \sum_{p, \overline{p}=0}^{\infty} \frac{\Gamma(1 + j_1 + m_1) \Gamma(1 + j_3 + m_3)}{p! \Gamma(-j_1 + m_1)} \Gamma(-j_1 + m_1 + p) \Gamma(1 + j_2 + m_2) \\
&\quad \times \frac{\Gamma(1 + j_1 + \overline{m}_1) \Gamma(1 + j_3 + \overline{m}_3)}{\overline{p}! \Gamma(-j_1 + \overline{m}_1) \Gamma(-j_2 - \overline{m}_2)} \Gamma(-j_1 + \overline{m}_1 + \overline{p}) \zeta^{-j_1} \overline{\zeta}^{-j_1} (\zeta - 1)^{-m_2 + p} (\overline{\zeta} - 1)^{-\overline{m}_2 + \overline{p}} \\
&= \frac{\tilde{C}(1 + j_i)}{\gamma(j_1 + j_2 + j_3 + 3 - \frac{k}{2})} \frac{\Gamma(1 + j_2 + m_2)}{\Gamma(-j_2 - \overline{m}_2)} \frac{\Gamma(1 + j_1 + m_1) \Gamma(1 + j_3 + \overline{m}_3)}{\Gamma(-j_1 - \overline{m}_1) \Gamma(-j_3 - m_3)} \zeta^{-m_1} (\zeta - 1)^{-m_2} \times c.c. \\
&\quad (4.36)
\end{aligned}$$

where we have used that, for any $a \in \mathbb{R}$,

$$F(-n, a, a|z) = (1 - z)^n. \quad (4.37)$$

Therefore

$$\mathcal{A}_3^{w=1} = \frac{\tilde{C}(1 + j_i)}{\gamma(j_1 + j_2 + j_3 + 3 - \frac{k}{2})} \frac{\Gamma(1 + j_1 + m_1)}{\Gamma(-j_1 - \overline{m}_1)} \frac{\Gamma(1 + j_2 + m_2)}{\Gamma(-j_2 - \overline{m}_2)} \frac{\Gamma(1 + j_3 + \overline{m}_3)}{\Gamma(-j_3 - m_3)}, \quad (4.38)$$

where, following the prescription in [4, 6], we have cancelled the ζ dependence in the four point function in order to get the one unit spectral flow three-point function

This result, worked out in the free field approach, agrees with the exact expression reported in [4] (with the obvious convention changes, see (2.53)). It has been obtained for one state in the discrete series. However it also holds for the continuous series. This may be seen by noting that the three-point function (4.38) does not change upon reflection of the vertex operator according to the quantum version of (2.35)¹⁰. Although the dependence of (4.38) on s is not obvious, it is exclusively contained in the factor $\gamma(s - k + 3)$ in the denominator. Therefore, the analytic continuation to non-integer s presents no difficulties. To our knowledge, this is the first independent verification of the exact result obtained in [4] for the one unit spectral flow three-point function for generic states.

4.3 Alternative method to compute spectral flow non-conserving amplitudes

In the previous section we computed a four-point function involving only $w = 0$ states. It is straightforward to see that the result still holds for states in arbitrary winding sectors as long as $\sum_i w_i = 0$. Even though this condition is consistent with the prescription in [4, 6], the natural way to introduce winding violation in the Coulomb gas approach is through a modification of the conservation laws. In this section we argue for an alternative procedure to compute winding number non-conserving correlators.

Recall that non-vanishing n -point functions must verify the following constraints [15]

$$\sum_{i=1}^N (j_i - m_i - w_i) - s = -1, \quad (4.39)$$

$$\sum_{i=1}^N (j_i - m_i) - s = -1, \quad (4.40)$$

$$\sum_{i=1}^N (j_i + \frac{k-2}{2} w_i) - s = -1. \quad (4.41)$$

It is possible to achieve $\sum_i w_i = \pm 1$ inserting the following auxiliary operators

$$\eta^-(\zeta) = e^{iv(\zeta)}, \quad (4.42)$$

$$\eta^+(\zeta) = e^{(k-2)u(\zeta)} e^{-i(k-1)v(\zeta)} e^{\sqrt{2(k-2)}\phi(\zeta)}. \quad (4.43)$$

These vertices correspond to operators with quantum numbers $j = \frac{k}{2} - 1, m = \pm \frac{k}{2}, w = \mp 1$ (see (2.29)). η^\mp can be thought of as identities in the $w = \mp 1$ sectors. This is emphasised by the fact that they have zero conformal dimension. Even though they have non-trivial OPE with physical vertex operators, namely

$$V_{j,m,\bar{m}}^w(z) e^{iv(\zeta)} \sim (z - \zeta)^{j-m} V_{j+\frac{k}{2}-1, m+\frac{k}{2}, w-1}(\zeta), \quad (4.44)$$

$$V_{j,m,\bar{m}}^w(z) e^{(k-2)u(\zeta)} e^{-i(k-1)v(\zeta)} e^{\sqrt{2(k-2)}\phi(\zeta)} \sim (z - \zeta)^{-j-m} V_{j+\frac{k}{2}-1, m-\frac{k}{2}, w+1}(\zeta), \quad (4.45)$$

the physical states on the left- and right-hand sides of these expressions can be identified using the equivalence between $\mathcal{D}_j^{\pm, w \mp 1}$ and $\mathcal{D}_{\frac{k}{2}-2-j}^{\mp, w}$. Moreover, the operators η^\pm commute with the currents up to a singular state. Notice the similarity of these properties and those of the spectral flow operator proposed in [6].

¹⁰This requires replacing the factor $1 + 2j$ in (2.35) by $\mathcal{R}(j) = [\rho(1 - 2j)\gamma(\rho(1 - 2j))]^{-1}$.

Since physical correlators should not depend upon the points where these operators are inserted, we can freely assume that they are located at infinity, consistently with a modification of the background charge in (4.39)–(4.41).

We now show that this prescription reproduces the expected result for the one unit spectral flow three-point function. Let us start by computing the four-point function

$$\mathcal{A}_4 = \langle V_{j_1 m_1 \bar{m}_1}^{w_1}(0) V_{j_2 m_2 \bar{m}_2}^{w_2}(1) V_{j_3 m_3 \bar{m}_3}^{w_3}(\infty) \eta^-(\zeta) \mathcal{S}^s \rangle, \quad (4.46)$$

which leads to the following conservation laws

$$s = \sum_i j_i + \frac{k}{2}, \quad (4.47)$$

$$\sum_i m_i + \frac{k}{2} = 0, \quad (4.48)$$

$$\sum_i w_i = 1. \quad (4.49)$$

The contribution from ϕ is not modified with respect to the spectral flow conserving case, i.e., it gives (3.8). The $u - v$ correlator now contributes

$$\zeta^{(-j_1+m_1)}(\zeta-1)^{(-j_2+m_2)} \bar{\zeta}^{(-j_1+\bar{m}_1)}(\bar{\zeta}-1)^{(-j_2+\bar{m}_2)} \prod_{i=1}^s |y_i|^{-2w_1} |1-y_i|^{-2w_2} \mathcal{P}^{-1} \partial_1 \cdots \partial_s \mathcal{Q} \bar{\mathcal{P}}^{-1} \partial_1 \cdots \partial_s \bar{\mathcal{Q}} \quad (4.50)$$

where \mathcal{P} and \mathcal{Q} are given by (3.10) and (4.7), respectively.

Putting everything together, it is easy to see that the dependence of the four-point function on ζ exactly cancels. Indeed, the quotient of determinants in (4.50) can be read from (4.23) in the case $m_1 = \bar{m}_1 = j_1$. In order to relax the highest-weight condition all the steps performed in the previous section go through to this case up to equation (4.35). Therefore the one unit spectral flow three-point function is obtained directly in this way, through well defined operators.

So far we have seen that the operator η^- reproduces the prescription presented in [6] and used in [4], in the particular case of one winding unit. Now we will show that this statement is general.

Let us assume that we are computing a correlator in the sector $w = N$. We need to evaluate expectation values of the form ¹¹

$$\mathcal{A}_3^{w=N} = \Gamma(-s) \left\langle \tilde{\mathcal{V}}_{j_1, m_1, \bar{m}_1}^{w_1}(0) \tilde{\mathcal{V}}_{j_2, m_2, \bar{m}_2}^{w_2}(1) \tilde{\mathcal{V}}_{j_3, m_3, \bar{m}_3}^{w_3}(+\infty) \prod_{i=1}^{N_+} \eta^+(\zeta_i^+) \prod_{l=1}^{N_-} \eta^-(\zeta_l^-) \mathcal{S}^s \right\rangle, \quad (4.51)$$

where $w = \sum_i w_i = N = N_- - N_+$.

The contribution from contractions of ϕ is the following

$$\prod_{l=1}^{N_+} |\zeta_l^+|^{-4j_1} |1-\zeta_l^+|^{-4j_2} \prod_{r < l} |\zeta_r^+ - \zeta_l^+|^{-4(k-2)} \prod_{i=1}^s |y_i|^{-4j_1 \rho} |1-y_i|^{-4j_2 \rho} \prod_{l=1}^{N_+} |\zeta_l^+ - y_i|^4 \prod_{i < j}^s |y_i - y_j|^{4\rho}, \quad (4.52)$$

¹¹Of course this three-point function vanishes for $|N| > 1$. Here we are interested in comparing the ζ -dependence with the prescription in [6]

whereas contractions of the $u - v$ fields give

$$\begin{aligned} & \prod_{m=1}^{N_-} (\zeta_m^-)^{-(j_1-m_1)} (1 - \zeta_m^-)^{-(j_2-m_2)} \prod_{m < r} (\zeta_m^- - \zeta_r^-) \prod_{l=1}^{N_+} (\zeta_l^+)^{(j_1-m_1)} (1 - \zeta_l^+)^{(j_2-m_2)} \\ & \times \prod_{l < n} (\zeta_l^+ - \zeta_n^+)^{2k-3} \times \prod_{l=1}^{N_+} \prod_{m=1}^{N_-} (\zeta_l^+ - \zeta_m^-)^{-(k-1)} \times \mathcal{P}^{-1} \partial_1 \dots \partial_s (\Lambda \mathcal{P}) \end{aligned} \quad (4.53)$$

times the antiholomorphic contribution, where

$$\mathcal{P} = \det \left[\prod_{l=1}^{N_+} (\zeta_l^+ - y_i)^{-k+2} (1 - y_i)^{-j_2+m_2} y_i^{j-1-j_1+m_1} \right] \quad (4.54)$$

and now

$$\Lambda = \prod_{i=1}^s \prod_{m=1}^{N_-} (\zeta_m^- - y_i) \prod_{l=1}^{N_+} (\zeta_l^+ - y_i)^{-1}. \quad (4.55)$$

We have omitted the powers involving w_i since they cancel.

Putting everything together we get

$$\begin{aligned} \mathcal{A}_3^{w=N} &= \Gamma(-s) \prod_{m=1}^{N_-} (\zeta_m^-)^{-j_1+m_1} (\bar{\zeta}_m^-)^{-j_1+\bar{m}_1} (1 - \zeta_m^-)^{-j_2+m_2} (1 - \bar{\zeta}_m^-)^{-j_2+\bar{m}_2} \prod_{m < r} (\zeta_m^- - \zeta_r^-) (\bar{\zeta}_m^- - \bar{\zeta}_r^-) \\ & \times \prod_{l=1}^{N_+} (\zeta_l^+)^{-j_1-m_1} (\bar{\zeta}_l^+)^{-j_1-\bar{m}_1} (1 - \zeta_l^+)^{-j_2-m_2} (1 - \bar{\zeta}_l^+)^{-j_2-\bar{m}_2} \prod_{l < n} (\zeta_l^+ - \zeta_n^+) (\bar{\zeta}_l^+ - \bar{\zeta}_n^+) \\ & \times \prod_{m=1}^{N_-} \prod_{l=1}^{N_+} (\zeta_l^+ - \zeta_m^-)^{-(k-1)} (\bar{\zeta}_l^+ - \bar{\zeta}_m^-)^{-(k-1)} \int [dy] \prod_{i=1}^s |y_i|^{-4j_1\rho} |1 - y_i|^{-4j_2\rho} \prod_{l=1}^{N_-} |\zeta_l^- - y_i|^4 \\ & \times \prod_{i < j}^s |y_i - y_j|^{4\rho} \mathcal{P}^{-1} \partial_1 \dots \partial_s (\Lambda \mathcal{P}) \times \bar{\mathcal{P}}^{-1} \bar{\partial}_1 \dots \bar{\partial}_s (\bar{\Lambda} \bar{\mathcal{P}}), \end{aligned} \quad (4.56)$$

which can be recast as

$$\begin{aligned} \mathcal{A}_3^{w=N} &= \prod_{m=1}^{N_-} (\zeta_m^-)^{m_1} (\bar{\zeta}_m^-)^{\bar{m}_1} (1 - \zeta_m^-)^{m_2} (1 - \bar{\zeta}_m^-)^{\bar{m}_2} \prod_{m < r} (\zeta_m^- - \zeta_r^-)^{k/2} (\bar{\zeta}_m^- - \bar{\zeta}_r^-)^{k/2} \\ & \times \prod_{l=1}^{N_+} (\zeta_l^+)^{-m_1} (\bar{\zeta}_l^+)^{-\bar{m}_1} (1 - \zeta_l^+)^{-m_2} (1 - \bar{\zeta}_l^+)^{-\bar{m}_2} \prod_{l < n} (\zeta_l^+ - \zeta_n^+)^{k/2} (\bar{\zeta}_l^+ - \bar{\zeta}_n^+)^{k/2} \\ & \times \prod_{l=1}^{N_+} \prod_{m=1}^{N_-} (\zeta_l^+ - \zeta_m^-)^{-k/2} (\bar{\zeta}_l^+ - \bar{\zeta}_m^-)^{-k/2} \times \mathcal{A}_{N+3}^{w=0}, \end{aligned} \quad (4.57)$$

with

$$\begin{aligned}
\mathcal{A}_{N+3}^{w=0} &= \Gamma(-s) \prod_{m=1}^{N_-} |\zeta_m^-|^{-2j_1} |1 - \zeta_m^-|^{-2j_2} \prod_{m < r} |\zeta_m^- - \zeta_r^-|^{-(k-2)} \prod_{l=1}^{N_+} |\zeta_l^+|^{-2j_1} |1 - \zeta_l^+|^{-2j_2} \\
&\times \prod_{l < n} (\zeta_l^+ - \zeta_n^+)^{-(k-2)} \prod_{m=1}^{N_-} \prod_{l=1}^{N_+} |\zeta_l^+ - \zeta_m^-|^{-(k-2)} \\
&\times \int [dy] \prod_{i=1}^s |y_i|^{-4j_1\rho} |1 - y_i|^{-4j_2\rho} \prod_{l=1}^{N_+} |\zeta_l^+ - y_i|^4 \prod_{i < j}^s |y_i - y_j|^{4\rho} \left[\frac{1}{\mathcal{P}} \partial_1 \dots \partial_s (\Lambda \mathcal{P}) \times \text{c.c.} \right].
\end{aligned} \tag{4.58}$$

We can now identify $\mathcal{A}_{N+3}^{w=0}$ with a $(N+3)$ -point function involving three physical states with winding number adding up to zero and N spectral flow operators, namely

$$\begin{aligned}
\mathcal{A}_{N+3}^{w=0} &= \Gamma(-s) \prod_{m=1}^{N_-} |\zeta_m^-|^{-2j_1} |1 - \zeta_m^-|^{-2j_2} \prod_{m < r} |\zeta_m^- - \zeta_r^-|^{-(k-2)} \prod_{l=1}^{N_+} |\zeta_l^+|^{-2j_1} |1 - \zeta_l^+|^{-2j_2} \\
&\times \prod_{l < n} |\zeta_l^+ - \zeta_n^+|^{-(k-2)} \prod_{m=1}^{N_-} \prod_{l=1}^{N_+} |\zeta_l^+ - \zeta_m^-|^{-(k-2)} \int [dy] \prod_{i=1}^s |y_i|^{-4j_1\rho} |1 - y_i|^{-4j_2\rho} \\
&\times \prod_{m=1}^{N_-} |\zeta_m^- - y_i|^2 \prod_{l=1}^{N_+} |\zeta_l^+ - y_i|^2 \prod_{i < j}^s |y_i - y_j|^{4\rho} \left[\frac{1}{\mathcal{Q}} \partial_1 \dots \partial_s \mathcal{Q} \times \text{c.c.} \right],
\end{aligned} \tag{4.59}$$

where

$$\mathcal{Q} = \Lambda \mathcal{P} = \det \left[\prod_{m=1}^{N_-} (\zeta_m^- - y_i) \prod_{l=1}^{N_+} (\zeta_l^+ - y_i)^{-(k-1)} (1 - y_i)^{-j_2+m_2} y_i^{j_1-1-j_1+m_1} \right]. \tag{4.60}$$

We thus see that the operators η^\pm produce the effect described in the previous section, since the ζ^\pm factors in $\mathcal{A}_3^{w=N}$ in (4.57) are precisely those that are to be cancelled ad-hoc according to the prescription in [6]. This shows that our approach completely agrees with the one suggested in [6], with the additional advantage that here it is implemented through well defined operators.

5 Summary and conclusions

We have proved that a proper treatment of the background charge method reproduces the generic three-point functions in the $SL(2, \mathbb{R})$ WZNW model. Indeed, we have found complete agreement with all previous exact results obtained from the bootstrap approach [4, 5, 6, 8]. Our work completes previous calculations performed in [15, 19] where the Coulomb gas formalism was used to compute spectral flow conserving three-point functions containing at least one state of the discrete highest-weight series or two highest-weight states in the spectral flow non-conserving case, respectively. Indeed, the highest-weight condition considered in [15, 19] allowed to simplify the computation of the $\beta - \gamma$ contribution to the three-point functions and it also permitted the use of the well known Dotsenko-Fateev integrals. However, while the analytic continuation to global descendant discrete states performed in [15] gives results in accordance with those of [4, 5], as shown in [20], the more general case had not been considered before.

Here we have been able to compute the ghost contribution for generic states. We showed that it can be expressed in terms of Schur polynomials and we solved the resulting new integrals of Dotsenko-Fateev type, namely we computed Aomoto integrals in the complex plane. Finally, we used monodromy invariance to properly perform the analytic continuation to non-integer number of screening operators. Actually, we proved that all spectral flow conserving three-point functions can be obtained acting with a suitable differential operator on a four-point-like function and thus showed that a monodromy invariant compatible analytic continuation is required. The techniques developed in Section 3 to deal with the spectral flow conserving three-point function were then applied to solve the one unit spectral flow three-point function in Section 4. In all cases we verified agreement with known exact results. Moreover, we proposed a novel method to compute winding number non-conserving n -point functions through well defined operators and we showed that it reproduces the prescription suggested in [6], with the advantage that ad-hoc cancellations are not required. In fact, applying this method to the case $n = 3$ we have shown that the result is independent of the insertion point of the auxiliary operator, which suggests that it is possible to compute the one unit spectral flow three-point function in the free field approach through a modification of the background charge.

Having realized that the ghost contribution to the amplitudes can be expressed in terms of Schur polynomials and the resolution of Aomoto integrals in the complex plane are interesting byproducts of our work. These are important ingredients of other related problems in CFT and they will certainly be elements of the Coulomb gas computation of higher-point functions, though in a more complex version. Actually one important simplification in the three-point function is the appearance of a minimal partition. Instead, Schur polynomials do not reduce to the elementary symmetric polynomial in the case of an extra insertion point, as we have seen in the computation of the four-point function in section 4. This implies that the multiple integrals appearing in the four-point function not only get more involved because of the extra insertion point but also because they include a more complicated Schur polynomial. Nevertheless, even though it might seem unlikely to obtain explicit expressions for generic four-point functions, given that they are not available in the simpler Liouville theory, our results provide a step forward towards the resolution of the factorization of AdS_3 four-point functions and the determination of unitarity and full consistency of the theory using Coulomb gas techniques.

Indeed, beyond the formal aspects regarding the explicit confirmation of the validity of the free field method and the eventual mathematical justification of the Coulomb gas formalism in this non-RCFT, other important applications of our work are related to string theory on AdS_3 . Even if closed expressions for higher than three-point functions cannot be found, we expect that the techniques we have developed will enable us to analyse the consistency of this theory studying the short distance limit of four-point functions. It might be argued that the background charge method appears unnecessarily complicated to achieve this task, given that the spacetime Lie and conformal symmetries allowed to resolve the problem exactly in the related H_3^+ model. In fact, factorization and crossing symmetry have been proved in [5, 8] using the relation between KZ and BPZ equations. Moreover, the analytic continuation of these results to string theory on AdS_3 was performed in [4], where the factorization of the four-point function of $w = 0$ states was shown to be consistent both with the Hilbert space proposed in [2] and with the winding violation pattern arising from the theory of $SL(2, \mathbb{R})$ representations. However, the inclusion of non-trivial spectral flow sectors simplifies in the formalism that we have presented here. In fact, whereas only one unit spectral flow vertex operators have been constructed in the x -basis, it is easy to include winding number in the m -basis. We believe that all these arguments justify the validation of the Coulomb gas approach up to the known exact correlators and the development of the new computational techniques which we have achieved in this article.

Finally, the extension of our results both to higher-point functions and to the supersymmetric case, started in [32], might be relevant to the scope of verifying the AdS_3/CFT_2 correspondence, another relevant

issue which has not been completed yet (see [33, 34, 35] for some recent work in this direction).

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6 Appendix: useful formulas

In this appendix we collect some of the formulas we have used in our computations.

1. Selberg integral and Dotsenko-Fateev formula.

The following integral was first derived by Selberg in [30]:

$$\begin{aligned} S_s(a, b, \rho) &= \int_0^1 dy_1 \cdots \int_0^1 dy_s \prod_{i=1}^s y_i^{a-1} (1-y_i)^{b-1} \prod_{i<j} |y_i - y_j|^{2\rho} \\ &= \prod_{i=0}^{s-1} \frac{\Gamma(a+i\rho)\Gamma(b+i\rho)\Gamma((i+1)\rho+1)}{\Gamma(a+b+(s+i-1)\rho)\Gamma(\rho+1)}. \end{aligned} \quad (6.1)$$

The extension of Selberg integral to the complex plane was carried out by Dotsenko and Fateev in [23]. They obtained the following result:

$$\begin{aligned} \mathcal{S}_s(a, b, \rho) &= \int \prod_{i=1}^n d^2 y_i \prod_{i=1}^s |y_i|^{2a-2} |1-y_i|^{2b-2} \prod_{i<j} |y_i - y_j|^{4\rho} \\ &= s! \pi^s \gamma(1-\rho)^s \prod_{i=1}^s \gamma(i\rho) \prod_{i=0}^{s-1} \gamma(a+i\rho) \gamma(b+i\rho) \gamma(1-a-b-(s-1+i)\rho). \end{aligned} \quad (6.2)$$

2. Aomoto integrals of order k .

In [21] Aomoto computed a family of integrals generalizing Selberg's. Aomoto's integral of order k is defined as

$$A_s^k(a, b, \rho) = \int_0^1 dy_1 \cdots \int_0^1 dy_s \alpha_k^s(y_1, \dots, y_s) \prod_{i=1}^s y_i^{a-1} (1-y_i)^{b-1} \prod_{i<j} |y_i - y_j|^{2\rho}, \quad (6.3)$$

where $\alpha_k^s(y_1, \dots, y_s)$ is the elementary symmetric polynomial of order k , i.e.,

$$\alpha_k^n(y_1, \dots, y_n) = \sum_{1 \leq j_1 < \dots < j_k \leq n} \prod_{i=1}^k y_{j_i} = \frac{1}{k!(n-k)!} \sum_{\sigma_n} \prod_{i=1}^k y_{\sigma_n(i)}, \quad (6.4)$$

and the last sum is made over the permutations of order n .

The following result was obtained in [21]:

$$A_s^k(a, b, \rho) = \binom{s}{k} \frac{\Gamma(\alpha + s)\Gamma(\alpha + \beta + 2s - k - 1)}{\Gamma(\alpha + s - k)\Gamma(\alpha + \beta + 2s - 1)} S_s(a, b, \rho), \quad (6.5)$$

where $\alpha = a/\rho$ and $\beta = b/\rho$.

Aomoto integrals can be conveniently arranged in a single expression. Let us consider the following integral:

$$A_s(a, b, \rho; z) = \int_0^1 dy_1 \cdots \int_0^1 dy_s \prod_{i=1}^s y_i^{\alpha-1} (1-y_i)^{\beta-1} (z-y_i) \prod_{i < j} |y_i - y_j|^{2\rho}. \quad (6.6)$$

Notice that $A_s(a, b, \rho; z)$ is an s -degree polynomial in the variable z . Using Newton identities, namely,

$$\prod_{i=1}^s (z - y_i) = \sum_{k=0}^s (-1)^k \alpha_k^s(y_1, \dots, y_s) z^{s-k} = \sum_{k=0}^s (-1)^{s-k} \alpha_{s-k}^s(y_1, \dots, y_s) z^k, \quad (6.7)$$

it is easy to see that

$$A_s(a, b, \rho; z) = \sum_{k=0}^s (-1)^k A_s^k(a, b, \rho) z^{s-k}, \quad (6.8)$$

i.e., Aomoto integral of order k is, up to a phase, the coefficient of the $(s-k)$ -degree term of $A_s(a, b, \rho; z)$. This can be more conveniently written in terms of monic Jacobi polynomials as

$$A_n(a, b, \rho; z) = \frac{(-1)^n}{2^n} S_n(a, b, \rho; z) \overline{P}_n^{\alpha-1, \beta-1}(1-2z), \quad (6.9)$$

where we have used the definitions collected below.

3. Jacobi polynomials.

Recall the following expression of Jacobi polynomials (see [36]):

$$\begin{aligned} P_n^{\alpha, \beta}(x) &= \frac{\Gamma(\alpha + n + 1)}{\Gamma(\alpha + 1)\Gamma(n + 1)} {}_2F_1\left(-n, \alpha + \beta + n + 1; \alpha + 1; \frac{1-x}{2}\right) \\ &= \frac{\Gamma(\alpha + n + 1)}{\Gamma(n + 1)\Gamma(\alpha + \beta + n + 1)} \sum_{k=0}^n \binom{n}{k} \frac{\Gamma(\alpha + \beta + n + k + 1)}{\Gamma(\alpha + k + 1)} \left(\frac{x-1}{2}\right)^k. \end{aligned} \quad (6.10)$$

Monic Jacobi polynomials read

$$\begin{aligned} \overline{P}_n^{\alpha, \beta}(x) &= 2^n \frac{\Gamma(\alpha + n + 1)\Gamma(\alpha + \beta + n + 1)}{\Gamma(\alpha + 1)\Gamma(\alpha + \beta + 2n + 1)} {}_2F_1\left(-n, \alpha + \beta + n + 1; \alpha + 1; \frac{1-x}{2}\right) \\ &= 2^n \frac{\Gamma(\alpha + n + 1)}{\Gamma(\alpha + \beta + 2n + 1)} \sum_{k=0}^n \binom{n}{k} \frac{\Gamma(\alpha + \beta + n + k + 1)}{\Gamma(\alpha + k + 1)} \left(\frac{x-1}{2}\right)^k \\ &= 2^n \frac{\Gamma(n + 1)\Gamma(\alpha + \beta + n + 1)}{\Gamma(\alpha + \beta + 2n + 1)} P_n^{\alpha, \beta}(x). \end{aligned} \quad (6.11)$$

4. Fukuda-Hosomichi integral.

The following double integral has been derived in [28]:

$$\begin{aligned} W(\alpha_i, \bar{\alpha}_i, \alpha'_i, \bar{\alpha}'_i, \sigma) &= \int d^2 z d^2 w \, z^{\alpha_1} (1-z)^{\alpha_2} \bar{z}^{\bar{\alpha}_1} (1-\bar{z})^{\bar{\alpha}_2} w^{\alpha'_1} (1-w)^{\alpha'_2} \bar{w}^{\bar{\alpha}'_1} (1-\bar{w})^{\bar{\alpha}'_2} |z-w|^{4\sigma} \\ &= \left(\frac{i}{2}\right)^2 \{C^{12}[\alpha_i, \alpha'_i] P^{12}[\bar{\alpha}_i, \bar{\alpha}'_i] + C^{21}[\alpha_i, \alpha'_i] P^{21}[\bar{\alpha}_i, \bar{\alpha}'_i]\}, \end{aligned} \quad (6.12)$$

where

$$C^{ab}[\alpha_i, \alpha'_i] = \frac{\Gamma(1 + \alpha_a + \alpha'_a - k') \Gamma(1 + \alpha_b + \alpha'_b - k')}{\Gamma(k' - \alpha_c - \alpha'_c)} G \left[\begin{matrix} \alpha'_a + 1, \alpha_b + 1, k' - \alpha_c - \alpha'_c \\ 1 - \alpha_c + \alpha'_a, \alpha_b - \alpha'_c + 1 \end{matrix} \right], \quad (6.13)$$

and

$$\left(\frac{i}{2}\right)^2 \begin{bmatrix} P^{12} \\ P^{21} \end{bmatrix} = A_\beta \begin{bmatrix} C^{23} \\ C^{32} \end{bmatrix} = A_\alpha^T \begin{bmatrix} C^{31} \\ C^{13} \end{bmatrix}, \quad (6.14)$$

with

$$G \left[\begin{matrix} a, b, c \\ e, f \end{matrix} \right] = \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(e)\Gamma(f)} {}_3F_2 \left[\begin{matrix} a, b, c \\ e, f \end{matrix} \middle| 1 \right],$$

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 + 1 &= k' = -2\sigma - 1, \\ \alpha'_1 + \alpha'_2 + \alpha'_3 + 1 &= k' = -2\sigma - 1, \end{aligned}$$

$$A_\alpha \begin{bmatrix} s(\alpha)s(\alpha') & -s(\alpha)s(\alpha' - k') \\ -s(\alpha')s(\alpha - k') & s(\alpha)s(\alpha') \end{bmatrix},$$

and $s(x) = \sin(\pi x)$.

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